A Complex number Z is defined to be an ordered pair (x,y) of real numbers x and y satisfying the following rules for addition and multiplication:

1) If $Z_1 = (x_1, y_1)$ and $Z_2 = (x_2, y_2)$ are two complex numbers then

$$Z_1 + Z_2 = (x_1 + x_2, y_1 + y_2) \quad \text{[Addition]}$$

2) $$Z_1 . Z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2) \quad \text{[multiplication]}$$

Two complex numbers $Z_1 = (x_1, y_1)$ and $Z_2 = (x_2, y_2)$ are said to be equal, written as $Z_1 = Z_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

The set of all complex numbers is denoted by $\mathbb{C}$. In $\mathbb{C}$, define two binary operations addition $(+)$ and multiplication $(.)$ of two complex numbers. With respect to these two operations, $\mathbb{C}$ forms a field. $(0,0)$ and $(1,0)$ are respectively the zero and the unity in $\mathbb{C}$.

The mapping $f: \mathbb{R} \to \mathbb{C}$ given by $f(x) = (x, 0)\ \forall x \in \mathbb{R}$ is a field isomorphism. So the real field $\mathbb{R}$ can be considered as a subfield of $\mathbb{C}$ (i.e any real number can be identified with the complex no. $(x,0)$. Conversely any complex no $(x,0)$ can be treated as real number x).

Complex numbers of the form $(x,0)$ are said to be purely real or real and those of the form $(0,y)$ are said to be purely imaginary whenever $y \neq 0$.

~~(0,0) is the only complex no. both real and purely imaginary~~. If $Z = (x, y)$, x is called the real part of Z and y is called imaginary part of Z.

The Complex number $(0,1)$ is written as i. Now $iy =$

$$(0,1)(y,0) = (0,y)$$

and $x+iy = (x,0)+(0,y) = (x,y)$

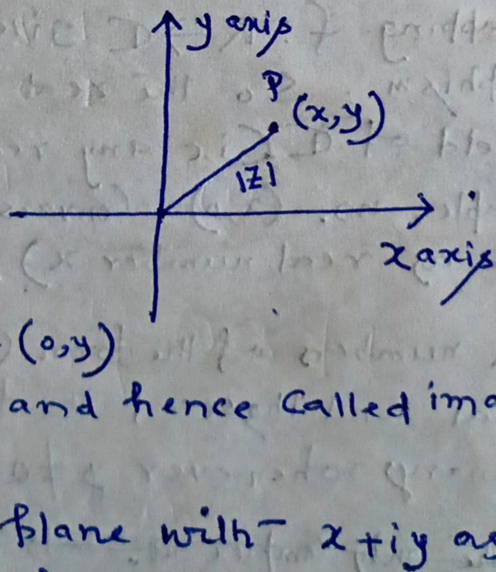
Also $i^2 = (0,1)(0,1) = (-1,0) = -1$.

A fundamental difference between $\mathbb{R}$ and $\mathbb{C}$ is that there is an order relation in $\mathbb{R}$ i.e if x and y are two distinct real numbers then either $x > y$ or $x < y$. But there is no such ~~orderetb~~ order relation in $\mathbb{C}$. Thus $z_1 > z_2$ or $z_1 < z_2$ will be meaningless in $\mathbb{C}$.

Geometric Representation of Complex numbers

Every Complex numbers(s) can be represented geometrically as a pt(s) in the xy-plane [geometric plane reffered to a pair of rectangular axes]. This plane is known as Complex plane or Argand plane.

Complex numbers of the form $(x,0)$ represents a point on x axis and hence x-axis is called real axis.



Complex numbers of the form $(0,y)$ represents a pt on y-axis and hence called imaginary axis.

Now we can visualize $\mathbb{C}$ as a plane with $x+iy$ as points (x,y) in $\mathbb{R}^2$ and we simply refer it as the complex plane or Argand plane.

We can extend the set of complex numbers $\mathbb{C}$ by introducing the symbol ∞ called complex infinity such that

1) $\infty + z = z + \infty = \infty \quad \forall z \in \mathbb{C}$.

2) $\infty . z = z . \infty = \infty \quad \forall z (\neq 0) \in \mathbb{C}$

3) $\infty . \infty = \infty$

4) $\frac{\infty}{z} = \infty, \quad \frac{z}{\infty} = 0 \quad \forall z \in \mathbb{C}$

5) $\frac{z}{0} = \infty \quad \forall z (\neq 0) \in \mathbb{C}$.

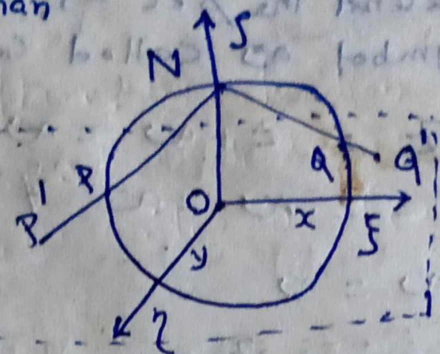
We do not define $\infty + \infty,\ 0.\infty,\ \frac{0}{0},\ \frac{\infty}{\infty}$.

We introduce an ideal point called the pt at infinity to represent the complex infinity ∞. The complex plane together with the pt at infinity is called the extended complex plane and is denoted by $\mathbb{C}_\infty$. Thus $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

For the complex no $z = x+iy$, the non negative real number $|z|$ is the modulus or absolute value of z, which is the distance of the point $P(x,y)$ from origin i,e $|z| = \sqrt{x^2+y^2}$.

Let $\mathbb{C}$ be the complex plane. Through the origin O construct a line perpendicular $\mathbb{C}$. Let this be ζ axis of a 3-dimensional euclidean space in which a pt has co-ordinates (ξ, η, ζ). The plane $\zeta = 0$ coincides with the complex plane $\mathbb{C}$ and that ξ and η axes are the x and y axes respectively. We consider a sphere S of unit radius with centred at origin (or $(0,0,0)$). Let ζ-axis meet the sphere at $N(0,0,1)$ (North pole). We associate to each point P of the sphere S, the point P′ on the complex plane where the line NP meets the complex plane and conversely to each point Q′ on the complex plane we associate the pt Q on the sphere where

The line NQ' meets the sphere other than N.



The only exception is the point N which has no corresponding point on the complex plane. We associate the pt p at infinity to N. Thus there is a one to one correspondance between the set of pts on the extended complex plane and the set of points on a sphere and this correspondance is called the stereographic projection. The sphere is called the Riemann sphere.

For Math.

Let $P(\xi,\eta,\zeta)$ be a point on S and $P'(x,y,0)$ be its corresponding point on the complex plane.

Then the three points $P(\xi,\eta,\zeta)$, $P'(x,y,0)$ and $N(0,0,1)$ are collinear. Hence

$$\frac{\xi-0}{x-0} = \frac{\eta-0}{y-0} = \frac{\zeta-1}{0-1}$$

or, $$\frac{\xi}{x} = \frac{\eta}{y} = \frac{\zeta-1}{-1} \quad \text{— (1)}$$

$$\Rightarrow \quad x = \frac{\xi}{1-\zeta} \text{ — (2)}, \quad y = \frac{\eta}{1-\zeta} \text{ — (3)}, \quad \zeta \neq 1$$

From (1), $$\left(\frac{\zeta-1}{-1}\right)^2 = \frac{\xi^2+\eta^2+(\zeta-1)^2}{x^2+y^2+(-1)^2}$$

$$(1-\zeta)^2 = \frac{2(1-\zeta)}{x^2+y^2+1} \quad [\text{Since } \xi^2+\eta^2+\zeta^2=1]$$

or, $1-\zeta = \dfrac{2}{x^2+y^2+1}$ [since $\zeta \neq 1$]

or $\zeta = 1 - \dfrac{2}{x^2+y^2+1} = \dfrac{x^2+y^2-1}{x^2+y^2+1}$

From (2), $\xi = x(1-\zeta) = \dfrac{2x}{x^2+y^2+1}$

From (3), $\eta = y(1-\zeta) = \dfrac{2y}{x^2+y^2+1}$.

Example. For each point of the following in $\mathbb{C}$, find the corresponding point of the Riemann sphere S: $-i$, $2+i$, i, $1+2i$

We know that if $z = x+iy$ be a point in $\mathbb{C}$, then the corresponding point (ξ, η, ζ) on the Riemann sphere S is given by

$$\xi = \frac{2x}{x^2+y^2+1}, \quad \eta = \frac{2y}{x^2+y^2+1}, \quad \zeta = \frac{x^2+y^2-1}{x^2+y^2+1}$$

For $z = 1+2i$

$$\xi = \frac{2.1}{1^2+2^2+1} = \frac{2}{6} = \frac{1}{3}$$

$$\eta = \frac{2.2}{6} = \frac{2}{3}$$

$$\zeta = \frac{1^2+2^2-1}{6} = \frac{4}{6} = \frac{2}{3}$$

The corresponding point on S is $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$.

Functions of a complex variable

Let S be a subset of complex plane. By a function f from S to $\mathbb{C}$, we mean a rule which assigns to each $z \in S$ a unique value $f(z) \in \mathbb{C}$. Symbolically, we write $w = f(z)$, $z \in S$

or, $\quad f : S \to \mathbb{C}$

We say f is a complex function of a complex variable. It is customary to locate $f(z) \in \mathbb{C}$ in another complex plane called the w-plane.

Let $z = x + iy$. Then every function $f(z)$ can be written as $f(z) = u(x,y) + iv(x,y)$ or $f = u + iv$. Here $u(x,y)$ and $v(x,y)$ are real functions of two variables and will be referred to as real and imaginary components of $f(z)$, respectively.

The definition of limit, continuity, uniform continuity is analogous to those in real analysis.

Let f be a function defined in some nbd of $z_0 \in \mathbb{C}$. The f is said to have limit w_0 at z_0 if, for given $\epsilon > 0$, there exist a $\delta > 0$ such that

$$|f(z) - w_0| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta.$$

Symbolically, we write $\lim\limits_{z \to z_0} f(z) = w_0$

or, formally this is also used in the form

$$f(z) = w_0 + \epsilon(z) \quad \forall z : 0 < |z - z_0| < \delta$$

with the condition that $\epsilon(z) \to 0$ as $z \to z_0$.

Let f be a ~~continuous~~ complex function defined on $D \subseteq \mathbb{C}$.
f is said to be continuous at a point $z_0 \in D$ if and only if for any $\epsilon > 0$, there exist $\delta > 0$ (δ depends on both ϵ and z) such that

$$|f(z) - f(z_0)| < \epsilon \text{ whenever } z \in D \text{ and } |z - z_0| < \delta.$$

[i.e $\lim_{z \to z_0} f(z)$ exists and equal to $f(z_0)$].

Th 1. Let $f(z) = u(x,y) + i v(x,y)$, $z = x+iy \in \mathbb{C}$, $z_0 = x_0 + iy_0$ and $l = l_1 + i l_2$

Then $\lim_{z \to z_0} f(z) = l$ iff $\lim_{(x,y) \to (x_0,y_0)} u(x,y) = l_1$ and $\lim_{(x,y) \to (x_0,y_0)} v(x,y) = l_2$

Th 2. If $\lim_{z \to z_0} f(z)$ exists then it is unique and f is bounded in some deleted nbd of z_0.

Proof: Let $\lim_{z \to z_0} f(z) = l$ (l is a complex no).

$\therefore$ For $\epsilon > 0$ $\exists$ $\delta > 0$ s.t

$$|f(z) - l| < \epsilon \quad \forall z: 0 < |z - z_0| < \delta$$

$$||f(z)| - l| < |f(z) - l| < \epsilon \quad \forall z: 0 < |z - z_0| < \delta$$

or $$|f(z)| < l + \epsilon \quad \forall z: 0 < |z - z_0| < \delta$$

So f is bounded in δ-nbd of z_0

Th. If $f = u + iv$ is continuous at $z_0 = x_0 + iy_0$ iff u and v both are continuous at (x_0, y_0)

Example 1. $f(z) = |z|^2 \ \forall z \in \mathbb{C}$

Show that f is continuous in the entire complex plane.

Ans: Let $f(z) = u(x,y) + iv(x,y)$, $z = x+iy \in \mathbb{C}$.

Then $u(x,y) = x^2+y^2$ and $v(x,y) = 0 \ \forall (x,y) \in \mathbb{R}^2$

Since u, v are continuous everywhere ~~f is continuous~~ (for all $(x,y) \in \mathbb{R}^2$), f is continuous for all $z \in \mathbb{C}$.

Example 2. $f(z) = \text{Re}\, z \ \forall z \in \mathbb{C}$ is continuous everywhere

Ans. Let $f(z) = u(x,y) + iv(x,y)$, $z = x+iy \in \mathbb{C}$

Then $u(x,y) = x$, $v(x,y) = 0 \ \forall (x,y) \in \mathbb{R}^2$

Since u, v are continuous everywhere, so f is continuous for all $z \in \mathbb{C}$.

Th Let $f, g : D \to \mathbb{C}$ be continuous at $z_0 \in D$. Then their sum $f+g$, product fg, quotient f/g where $g(z_0) \neq 0$, $|f|$, kf (k is const) are continuous at z_0.

Th Let $f : D \to \mathbb{C}$ ($D \subseteq \mathbb{C}$) is continuous at $z_0 \in D$ iff $\lim_{n\to\infty} f(z_n) = f(z_0)$ for every sequence $\{z_n\}_n$ in D converges to $z_0 \in D$.

Differentiability of complex functions

A complex ~~number~~ function f defined on a open set D is said to be differentiable at an interior point z_0 of D if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists.} \quad \text{—— (1)}$$

The value of the limit (if exists), denoted by $f'(z_0)$, is called the derivative of f at z_0.

If $z = z_0 + h$, h, a complex, then (1) is ~~equal~~ equivalently written as

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

f is said to be differentiable on D if it is differentiable at every pt of D.

Example $\quad f(z) = z^n$ ($n \geq 1$ is an integer) $\forall z \in \mathbb{C}$

Let $z_0 \in \mathbb{C}$

For $z \neq z_0$

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{z^n - z_0^n}{z - z_0} = z^{n-1} + z^{n-2} z_0 + \cdots + z_0^{n-1}$$

So $\displaystyle \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = n z_0^{n-1}$

So $f'(z_0) = n z_0^{n-1} \quad \forall z_0 \in \mathbb{C}$

Hence $f(z) = z^n$ is differentiable in the entire complex plane and $f'(z) = n z^{n-1} \quad \forall z \in \mathbb{C}$.

Th If f is differentiable at z_0, then it is continuous at z_0.

Proof: Let f be differentiable at z_0

Then $\displaystyle \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists and the value of the limit is denoted by $f'(z_0)$

For $z \neq z_0$

$$f(z)-f(z_0) = \frac{f(z)-f(z_0)}{z-z_0}(z-z_0)$$

So $\lim\limits_{z \to z_0} (f(z)-f(z_0)) = f'(z_0) \times 0 = 0$.

So $\lim\limits_{z \to z_0} f(z) = f(z_0)$ and hence f is continuous at z_0.

The converse of the above theorem is not true which follows from the following ~~table~~ example

Let $f(z) = |z| \ \forall z \in \mathbb{C}$

Now $\lim\limits_{z \to 0} f(z) = \lim\limits_{z \to 0} |z| = 0 = f(0)$

So f is continuous at 0

For $z \neq 0$

$$\frac{f(z)-f(0)}{z-0} = \frac{|z|}{z}.$$

Let $z(=x+iy) \to 0$ along the +ve part of real axis (then $y \doteq 0$, $x \to 0+$)

Then $\lim\limits_{z \to 0} \frac{|z|}{z} = \lim\limits_{x \to 0+} \frac{x}{x} = 1$

If $z \to 0$ along the -ve part of real axis (then $y \doteq 0$, $x \to 0^-$)

$\therefore \lim\limits_{z \to 0} \frac{|z|}{z} = \lim\limits_{x \to 0^-} \left(-\frac{x}{x}\right) = -1$

So $\lim\limits_{z \to 0} \dfrac{|z|}{z}$ does not exist.

So f is not differentiable at z_0.

<u>Note</u>: The function $f(z) = |z|$ is continuous everywhere but differentiable nowhere.

<u>Th</u>. If f, g are differentiable at z_0, then their sum $f+g$, product fg, quotient f/g where $g(z_0) \neq 0$, kf where k is constant are differentiable at z_0 and

$$(f \pm g)' = f' \pm g'$$
$$(fg)' = fg' + f'g$$
$$(f/g)' = \frac{gf' - fg'}{g^2}$$
$$(kf)' = kf'.$$

<u>Note</u>: Differentiability of f at a point does not imply the differentiability of $|f|$ at that point.

The function f defined by $f(z) = z$ is differentiable everywhere but $|f(z)| = |z|$ is nowhere differentiable.

<u>Note</u>: $\operatorname{Re} z$, $\operatorname{Im} z$, $|z|$, $\bar{z}$ are continuous everywhere but nowhere differentiable.

Analytic function

A function f is said to be analytic at at point $z = \alpha$ if f is differentiable at every point of some neighbourhood of α.

f is said to be analytic on a set S if f is differentiable at every pt of some open set containing S (i.e f is analytic at every pt of S)

Note: The function f is defined by $f(z) = |z|^2$ is ~~every~~ no where differentiable except at the origin

So f is not analytic at $z = 0$

$\therefore$ Differentiability at a pt does not imply that the function is analytic at that point.

Cauchy - Riemann's Equations

A complex function of complex variable f can be written as $f = u + iv$ where u, v are real valued functions of two real variables x & y.

i.e. if $z = x + iy$, $f(z) = u(x,y) + iv(x,y)$

The partial derivatives of u with-respect to x at (a,b), denoted by $u_x(a,b)$ if it exists, is defined by

$$\lim_{h \to 0} \frac{u(a+h, b) - u(a,b)}{h}$$ [here h is a real ~~variable~~ number]

The partial derivative of u with-respect to y at (a,b), denoted by $u_y(a,b)$ and if it exists, is defined to be

$$\lim_{h \to 0} \frac{u(a, b+h) - u(a,b)}{h}$$ [here h is a real number]

Similarly we can define $\frac{\partial v}{\partial x}$ or v_x and v_y.

The two equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

are called Couchy - Riemann's equations.

$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$ and $\frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)$ is are called

Couchy - Riemann (briefly C-R) equations at z_0.

The 1st order partial derivatives of u with respect to x at (x_0, y_0),

if it exists, denoted by $\frac{\partial u}{\partial x}(x_0, y_0)$ or $\left.\frac{\partial u}{\partial x}\right|_{(x_0, y_0)}$ and is defined by

or $u_x(x_0, y_0) =$

$$\lim_{h \to 0} \frac{u(x_0+h, y_0) - u(x_0, y_0)}{h} \quad [h \text{ is real number}]$$

$$\text{or } \lim_{x \to x_0} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0}$$

Illy $u_y(x_0, y_0)$, $v_x(x_0, y_0)$, $v_y(x_0, y_0)$ is defined.

Th. A necessary condition for the differentiability of the function $f = u + iv$ at a pt $z_0 = x_0 + iy_0$ is that the 1st order partial derivatives of u & v exist and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at the pt (x_0, y_0) i.e $\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$ and $\frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)$

[In other words if f is differentiable at pt $z_0 = x_0 + iy_0$ then C-R equations are satisfied at (x_0, y_0).]

Proof Let the function $f = u + iv$ be differentiable at the pt $z_0 = (= x_0 + iy_0)$. Then $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists and the value of the limit, $f'(z_0)$ is finite and is unique whatever path we adopt to let $z \to z_0$.

Now for $z (= x + iy) \neq z_0$

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{u(x,y) + iv(x,y) - \{u(x_0,y_0) + iv(x_0,y_0)\}}{(x+iy) - (x_0 + iy_0)}$$

$$= \frac{\{u(x,y) - u(x_0,y_0)\} + i\{v(x,y) - v(x_0,y_0)\}}{(x - x_0) + i(y - y_0)}$$

Suppose $z \to z_0$ along a line parallel to the real axis.

Then $y = y_0$, $x \to x_0$

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{x \to x_0} \left\{ \frac{u(x,y_0) - u(x_0,y_0)}{x - x_0} + i\, \frac{v(x,y_0) - v(x_0,y_0)}{x - x_0} \right\}$$

Since $f'(z_0)$ exists both the 1st order partial derivatives u_x, v_x exists at (x_0, y_0) and

$$f'(z_0) = u_x + i v_x \text{ at } (x_0, y_0) \quad \text{——— (1)}$$

Let $z \to z_0$ along a line $\parallel$ to the imaginary axis, then $x = x_0, y \to y_0$

So, $$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{y \to y_0} \left\{ \frac{u(x_0, y) - u(x_0, y_0)}{i(y - y_0)} + \frac{v(x_0, y) - v(x_0, y_0)}{y - y_0} \right\}$$

Since $f'(z_0)$ exists, u_y, v_y exist at (x_0, y_0)

Hence $$f'(z_0) = \frac{1}{i} u_y + v_y \text{ at } (x_0, y_0)$$
$$= -i u_y + v_y \text{ at } (x_0, y_0) \quad \text{—— (2)}$$

From (1) and (2), we get
$$u_x + i v_x = -i u_y + v_y \text{ at } (x_0, y_0)$$

Equating the real and imaginary parts
$$u_x = v_y \text{ at } (x_0, y_0)$$
and $$u_y = -v_x \text{ at } (x_0, y_0)$$

Hence the theorem

Ex (1) Let $$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

Let $f(z) = u(x, y) + i v(x, y) \ \forall z = x + iy \in \mathbb{C}$.

Then $$u(x, y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

and $$v(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Then $u_x(0,0) = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{x}{x} = 1$

$u_y(0,0) = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{-y}{y} = -1$

$v_x(0,0) = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0} \frac{x}{x} = 1$

$v_y(0,0) = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0} \frac{y}{y} = 1$

So $u_x = v_y$ at $(0,0)$ and

$u_y = -v_x$ at $(0,0)$

Hence C-R equations are satisfied at $(0,0)$

Now for $z = x+iy \neq 0+i0$

$$\frac{f(z)-f(0)}{z-0} = \frac{\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}}{x+iy}$$

$$= \frac{(x^3-y^3)+i(x^3+y^3)}{(x^2+y^2)(x+iy)}$$

Let $(x,y)\to(0,0)$ along the path $y=mx$

$$\lim_{(x,y)\to(0,0)} \frac{(x^3-y^3)+i(x^3+y^3)}{(x+iy)(x^2+y^2)} = \lim_{(x,y)\to(0,0)} \frac{(1-m^3)+i(1+m^3)}{(1+m^2)(1+im)}$$

which is different for different values of m.

Hence $\lim_{z \to 0} \frac{f(z)-f(0)}{z-0}$ i.e $f'(0)$ does not exist

So the function f is not differentiable at origin eventhrough they satisfy C-R equations there.

Ex 2. $f(z) = \begin{cases} \frac{xy^2 (x+iy)}{x^2+y^4}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

$u(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

$v(x,y) = \begin{cases} \frac{xy^3}{x^2+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

$u_x(0,0) = \lim_{x \to 0} \frac{u(x,0)-u(0,0)}{x} = 0$

Illly, $u_y(0,0) = v_x(0,0) = v_y(0,0) = 0$.

Hence C-R equations are satisfied at $(0,0)$.

Now for $z = x+iy \neq 0+i0$,

$\frac{f(z)-f(0)}{z-0} = \frac{xy^2(x+iy)}{(x^2+y^4)(x+iy)} \quad (x,y) \neq (0,0)$

$= \frac{xy^2}{x^2+y^4}, \quad (x,y) \neq (0,0)$

Let $(x,y) \to (0,0)$ along the path $x = my^2$

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{(x,y)\to(0,0)} \frac{m}{m^2+1} = \frac{m}{m^2+1}$$ which is different for different values of m.

Hence $\lim_{z \to 0} \frac{f(z)-f(0)}{z-0}$ i.e $f'(0)$ does not exist

EX 3.

$$f(z) = \begin{cases} \frac{(\bar{z})^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

$$f(z) = \frac{(\bar{z})^3}{|z|^2} = \frac{(x-iy)^3}{x^2+y^2} = \frac{x^3 - 3x^2(iy) + 3x(-y^2) - (-iy^3)}{x^2+y^2}$$

$$= \frac{(x^3 - 3xy^2) + i(y^3 - 3x^2y)}{x^2+y^2}$$

$$u(x,y) = \begin{cases} \frac{x^3 - 3xy^2}{x^2+y^2}, & (x,y) \neq 0 \\ 0, & (x,y) = 0 \end{cases}$$

$$v(x,y) = \begin{cases} \frac{y^3 - 3x^2y}{x^2+y^2}, & (x,y) \neq 0 \\ 0, & (x,y) = 0. \end{cases}$$

$$u_x(0,0) = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0} \frac{x^3}{x^3} = 1$$

$$u_y(0,0) = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y\to 0} \frac{0}{y^2} = 0$$

$$v_x(0,0) = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x}$$

$$= \lim_{x \to 0} \frac{0}{x} = 0$$

$$v_y(0,0) = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y}$$

$$= \lim_{y \to 0} \frac{y^3}{y^3} = 1.$$

$u_x(0,0) = v_y(0,0)$ & $u_y(0,0) = -v_x(0,0)$

Hence C-R equations are satisfied

Now for $z = x+iy \neq (0+i0)$

$$\frac{f(z) - f(0)}{z - 0} = \frac{(x^3 - 3xy^2) + i(y^3 - 3x^2y)}{(x^2+y^2)(x+iy)}$$

Let $(x,y) \to (0,0)$ along the path $y = mx$

$$\lim_{(x,y) \to (0,0)} \frac{(x^3 - 3xy^2) + i(y^3 - 3x^2y)}{(x^2+y^2)(x+iy)}$$

$$= \lim_{(x,y) \to (0,0)} \frac{(1-3m^2) + i(m^3 - 3m)}{(1+m^2)(1+im)}$$

$$= \frac{(1-3m^2) + i(m^3 - 3m)}{(1+m^2)(1+im)}$$ which is different for different values of m.

Hence $f'(0)$ does not exist

Suppose $f = u + iv$ is defined in some nbd of z_0 and that u_x, u_y, v_x, v_y are continuous at z_0 and u & v satisfy C-R equations at z_0. Then f is differentiable at z_0.

Proof Let $f = u + iv$ be defined in some neighbourhood $N(z_0)$ of z_0 and u_x, u_y, v_x & v_y are continuous at z_0 and u & v satisfy C-R equations at $z_0 = x_0 + iy_0$ i.e., $u_x = v_y$ and $u_y = -v_x$ at (x_0, y_0).

Now for $h = h_1 + i h_2 \neq 0$ s.t z_0+h in $N(z_0)$, we have

$$\frac{f(z_0+h) - f(z_0)}{h} = \frac{u(x_0+h_1, y_0+h_2) + iv(x_0+h_1, y_0+h_2) - u(x_0,y_0) - iv(x_0,y_0)}{h}$$

$$= \frac{1}{h}\left[\left\{u(x_0+h_1, y_0+h_2) - u(x_0,y_0)\right\} + i\left\{v(x_0+h_1, y_0+h_2) - v(x_0,y_0)\right\}\right] \quad \text{--- (1)}$$

Now $u(x_0+h_1, y_0+h_2) - u(x_0,y_0)$

$$= \left\{u(x_0+h_1, y_0+h_2) - u(x_0, y_0+h_2)\right\} + \left\{u(x_0, y_0+h_2) - u(x_0,y_0)\right\}$$

$$= \left\{u_x(x_0,y_0) + \epsilon_1\right\} h_1 + \left\{u_y(x_0,y_0) + \epsilon_2\right\} h_2$$ [Since u_x, u_y are continuous at (x_0,y_0)]

where $\epsilon_1 = \epsilon_1(h_1, h_2) \to 0$ as $(h_1,h_2) \to (0,0)$

and $\epsilon_2 = \epsilon_2(h_1,h_2) \to 0$ as $(h_1,h_2) \to (0,0)$.

Similarly, $v(x_0+h_1, y_0+h_2) - v(x_0,y_0) = \{v_x(x_0,y_0)+\eta_1\}h_1 + \{v_y(x_0,y_0)+\eta_2\}h_2$ ---- (3) where $\eta_1 = \eta_1(h_1,h_2) \to 0$ as $(h_1,h_2)\to(0,0)$ and $\eta_2 = \eta_2(h_1,h_2) \to 0$ as $(h_1,h_2)\to(0,0)$.

From (1), (2) and (3),

$$\frac{f(z_0+h)-f(z_0)}{h} = \frac{1}{h}\Big[\{u_x(x_0,y_0)+\epsilon_1\}h_1 + \{u_y(x_0,y_0)+\epsilon_2\}h_2 + i\{v_x(x_0,y_0)+\eta_1\}h_1 + i\{v_y(x_0,y_0)+\eta_2\}h_2\Big]$$

$$= \frac{1}{h}\Big[\{u_x(x_0,y_0)+i v_x(x_0,y_0)\}h_1 + \{u_y(x_0,y_0)+i v_y(x_0,y_0)\}h_2 + (\epsilon_1+i\eta_1)h_1 + (\epsilon_2+i\eta_2)h_2\Big]$$

$$= \frac{1}{h}\Big[\{u_x(x_0,y_0)+i v_x(x_0,y_0)\}h_1 + \{-v_x(x_0,y_0)+i u_x(x_0,y_0)\}h_2 + (\epsilon_1+i\eta_1)h_1 + (\epsilon_2+i\eta_2)h_2\Big]$$

(using C-R equations)

$$= u_x(x_0,y_0)+i v_x(x_0,y_0) + (\epsilon_1+i\eta_1)\frac{h_1}{h} + (\epsilon_2+i\eta_2)\frac{h_2}{h} \quad \text{—— (4)}$$

Now, $\left|\frac{h_1}{h}\right| \le 1$, $\left|\frac{h_2}{h}\right| \le 1$ and $(\epsilon_1+i\eta_1)\to 0$ as $(h_1,h_2)\to(0,0)$
$(\epsilon_2+i\eta_2)\to 0$ as $(h_1,h_2)\to(0,0)$

So taking limit as $h\to 0$ in (4), we get

$$\frac{f(z_0+h)-f(z_0)}{h} \to u_x(x_0,y_0)+i v_x(x_0,y_0) \text{ as } h\to 0.$$

This shows that f is differentiable at z_0 and

$$f'(z_0) = u_x(x_0,y_0)+i v_x(x_0,y_0).$$

Theorem: A real valued function of complex variable either has derivative zero or the derivative does not exist.

Proof: Let f be a real valued function of complex variable.
Suppose the derivative of f at a point z_0 exists. Then

$\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$ exists and the value of the limit, $f'(z_0)$ is finite and the value of the limit, $f'(z_0)$ is finite and is unique whatever path we adopt to let $h \to 0$ where $h = h_1 + i h_2$.

Suppose $h \to 0$ along the real axis. Then $h_2 = 0$ and $h_1 \to 0$

and $f'(z_0) = \lim_{h_1 \to 0} \frac{f(z_0+h_1) - f(z_0)}{h_1}$ is a real number — (1)

Let $h \to 0$ along the imaginary axis. Then $h_1 = 0$ and $h_2 \to 0$.

$$\text{and } f'(z_0) = \lim_{h_2 \to 0} \frac{f(z_0+ih_2) - f(z_0)}{ih_2}$$

$$= \frac{1}{i} \lim_{h_2 \to 0} \frac{f(z_0+ih_2) - f(z_0)}{h_2}$$

$$= \frac{1}{i} \cdot \text{a real number}$$

$= $ a purely imaginary number. — (2)

From (1) and (2) we have $f'(z_0) = 0$

So f has either derivative zero or the derivative does not exist

EX Show that the following functions are nowhere analytic

① $f(z) = x^2 + iy^2$ ② $f(z) = xy^2 + 2 + iy$

③ $f(z) = e^x(\cos y - i \sin y)$

① Let $f(z) = u(x,y) + iv(x,y)$ $\forall z = (x+iy)$

Then $u(x,y) = x^2$ & $v(x,y) = y^2$ $\forall (x,y) \in \mathbb{R}^2$.

Then $u_x = 2x$, $u_y = 0$, $v_x = 0$, $v_y = 2y$ $\forall (x,y)$

The functions u_x, u_y, v_x, v_y are continuous everywhere

Now C-R equ^ns:

$u_x = v_y$, $u_y = -v_x$ $\Rightarrow$ $2x = 2y$
$\Rightarrow x = y$

So C-R equations are satisfied only when $x = y$

Thus f is diff only at those points for which $x = y$ and nowhere else.

Since every nbd of only point on the line $y = x$ will contain pts at which f is not differentiable, the function f is not analytic.

(2) Let $f(z) = u(x,y) + i v(x,y)$ $\forall z = x+iy \in \mathbb{C}$.

Then $u(x,y) = xy^2 + 2$, $v(x,y) = y$ $\forall (x,y) \in \mathbb{R}^2$.

$u_x = y^2$, $u_y = 2xy$, $v_x = 0$, $v_y = 1$

u_x, u_y, v_x, v_y are continuous for all x, y.

C-R equ^ns are as follows:

$$u_x = v_y \;;\; u_y = -v_x$$

$$\Rightarrow y^2 = 1 \;;\; 2xy = 0$$

$$\Rightarrow y = \pm 1 \;;\; x = 0.$$

So C-R equ^ns are satisfied only at $(0,+1)$ & $(0,-1)$

So f is differentiable at $(0,1)$ and $(0,-1)$ and nowhere ~~analytic~~ else.

So f is nowhere analytic

(3) Let $f(z) = u(x,y) + i v(x,y)$ $\forall z = \cancel{x+iy}\ x+iy \in \mathbb{C}$.

Then $u(x,y) = e^x \cos y$ and $v(x,y) = -e^x \sin y$ $\forall (x,y) \in \mathbb{R}^2$

$$u_x = e^x \cos y, \quad u_y = -e^x \sin y$$

$$v_x = -e^x \sin y \;\&\; v_y = -e^x \cos y \quad \forall (x,y) \in \mathbb{R}^2$$

u_x, u_y, v_x, v_y are continuous for all $(x,y) \in \mathbb{R}^2$.

C-R equ^ns are as follows:

$$u_x = v_y \;\&\; v_x = -u_y$$

$$\Rightarrow e^x \cos y = 0 \;\&\; e^x \sin y = 0$$

$$\Rightarrow \cos y = 0 \;\&\; \sin y = 0$$

So C-R equations are not satisfied at any pt in $\mathbb{R}^2$.

So f is not differentiable at any pt.

Consequently f is nowhere analytic.

Domain or Region : open set, Connected set

Result 1: Suppose f is analytic in a region G

Then ① if $f'(z) = 0$ in G then f is constant in G.

② if any one of $\operatorname{Re} f$, $\operatorname{Im} f$, $|f|$ is constant in G then f is constant in G.

Proof ① Let $f = u + iv$ is analytic in a region G and $f'(z) = 0$ in G.

Then for all $z \in G$ i.e for all $(x,y) \in G$.

$$f'(z) = u_x(x,y) + i v_x(x,y)$$
$$= v_y(x,y) - i u_y(x,y)$$

$f'(z) = 0$ in $G \Rightarrow u_x = v_y = v_x = u_y = 0 \ \forall (x,y) \in G$.

$u_x = 0 \Rightarrow u$ is free from x

$u_y = 0 \Rightarrow u$ is free from y

Hence u is Constant

Similarly v is Constant

Hence $f = u + iv$ is Constant in G.

---

(2) Let $Re f = u$ is Const in G.

Then $u_x = u_y = 0$ in G

Since f is analytic in G, f satisfies C-R equations in G

Then $v_y = v_x = 0$ in G

Now $v_y = 0 \Rightarrow v$ is free from y

$v_x = 0 \Rightarrow v$ is free from x

$\therefore$ $v(x,y)$ is Constant in G.

Hence $f = u + iv$ is Constant in G.

Similarly ~~if~~ we can show that if $Im f = v$ is Const in G then f is Constant in G.

Let $|f|$ ~~is~~ be constant in G and $|f| = k \ \forall z \in G$ (k is Const)

If $k = 0$, then $f = 0$ in G.

So, let $k \neq 0$

Then $u^2 + v^2 = k^2$

Since f is analytic in G, differentiating partially w.r.t x and y, we get —

$$u u_x + v v_x = 0 \quad \text{—— (1)}$$

& $$u u_y + v v_y = 0 \quad \text{—— (2)}$$

By C-R equations (1) & (2) become

$$u u_x - v u_y = 0$$
$$u u_y + v u_x = 0$$

Squaring & adding, we get

$$(u^2+v^2)(u_x^2+u_y^2) = 0$$

or $K^2 (u_x^2+u_y^2) = 0$

or $u_x^2+u_y^2 = 0$

or $K^2 |f'(z)|^2 = 0$

or $|f'(z)| = 0$ (Since $K \neq 0$)

or $f'(z) = 0$

Hence by (1) f is const in G.

Result 2. Suppose f and g are analytic in a region G such that $f'(z) = g'(z)$ in G. Then $f-g$ is constant in G.

Proof: Let $\phi(z) = f(z) - g(z)$ $\forall z \in G$.

Since f & g are analytic in G, $\phi'(z)$ exist in G

and $\phi'(z) = f'(z) - g'(z) = 0$ $\forall z \in G$

Then ϕ ~~is~~ $= f-g$ is Constant in G.

[Note: ~~The function $f(z) = |z|$~~,

(1) f is analytic, real valued function of complex variable then f is constant function.

(2) f is analytic, f takes only purely imaginary values, then f is constant function.

Note f is differentiable on an open set D.

Then for each z, $f'(z)$ is finite

So we can define a function $f_1 : D \to \mathbb{C}$ by $f_1(z) = f'(z)$

If f_1 is differentiable then f is twice differentiable

Continuining, a differentiable function f be such that each successive derivative is again differentiable then f is called infinitely differentiable

Result: If f is analytic in a region G, then f is infinitely differentiable in G.

Harmonic functions.

A real valued function h of two variables x, y is said to be harmonic in a region G if it has second order continuous partial derivatives and ~~satisfy~~ satisfies the Laplace equ^n $h_{xx} + h_{yy} = 0$ in G.

Ex. Verify that the function $h(x,y) = 3x^2y - y^3 + 4$ is harmonic in $\mathbb{C}$.

Proof $h(x,y) = 3x^2y - y^3 + 4$.

$h_x = 6xy$, $h_y = 3x^2 - 3y^2$, $h_{xy} = 6x = h_{yx}$.

$h_{xx} = 6y$, $h_{yy} = -6y$.

$h_{xx}, h_{yy}, h_{xx}, h_{yx}$ are continuous and $h_{xx} + h_{yy} = 0$ in $\mathbb{C}$.

Hence ~~f~~ h is harmonic in $\mathbb{C}$.

Theorem If ~~a~~ $f = u + iv$ is analytic in a region G, then u and v are harmonic in G.

Proof: Let $f = u + iv$ be analytic in a region G.

Then f is infinitely differentiable in G.
So u, v have continuous second order partial derivatives in G, and so $u_{xy} = u_{yx}$, $v_{xy} = v_{yx}$ in G.

Since f is analytic in G, C-R equations :

$$u_x = v_y \quad \text{--- (1)}$$
$$u_y = -v_x \quad \text{--- (2)}$$
are satisfied in G.

Differentiating (1) w.r.to x and differentiating (2) w.r.to y and then adding, we get —

$$u_{xx} + u_{yy} = v_{xy} - v_{yx} = 0$$

So u satisfies Laplace's eq^n in G.

Again differentiating (1) w.r.to y and differentiating (2) w.r.to x and then adding, we get —

$$v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0$$

So v satisfies Laplace's eq^n in G.

Hence u & v are harmonic in G.

Note: If $f = u + iv$ be analytic in a region G, then u, v satisfy Laplace's eq^n in G.

Def^n: If two harmonic functions u & v satisfy C-R eq^{n}s : $u_x = v_y$ and $u_y = -v_x$ in a region G, then v is called conjugate harmonic of u in G.

Th A function $f = u+iv$ is analytic in a region G if and only if v is conjugate harmonic of u.

Proof: Let $f = u+iv$ be analytic in a region G

Then u, v are harmonic in G.

Also C-R eqns: $u_x = v_y$, $u_y = -v_x$ are satisfied.

Hence v is harmonic conjugate of u.

Conversely, suppose that v is conjugate harmonic of u.

Since u, v are harmonic in G, they have continuous second order partial derivatives in G.

So in particular u, v have continuous first order partial derivatives in G.

Now v is conjugate harmonic of u.

So C-R quations: $\left.\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right\}$ are satisfied in G.

Hence $f = u+iv$ is analytic.

~~Milne-Thompson Method~~

~~To construct an analytic function whose real & imaginary components are given.~~

Note: Let u be harmonic in some nbd of a pt. Then $\exists$ a conjugate harmonic function v defined in the nbd such that $f = u + iv$ is analytic in that nbd.

1. Determine a function $v(x,y)$ (conjugate harmonic of u) such that the function $f = u + iv$ is analytic where

i) $u(x,y) = 2xy + 2x$ ii) $= e^x \cos y$

iii) $= e^x(\cos y + y)$ iii) $= -x^3 + 3xy^2 + 2y + 1$

i) $u(x,y) = 2xy + 2x$

$u_x = 2y + 2$, $u_y = 2x$.

$u_{xx} = 0$, $u_{yx} = 2 = u_{xy}$, $u_{yy} = 0$

u_{xx}, u_{yy}, u_{xy}, u_{yx} are all continuous and $u_{xx} + u_{yy} = 0$ $\forall (x,y) \in \mathbb{R}^2$

So u is harmonic.

Hence it is possible to construct an analytic function $f = u + iv$.

Since f is analytic, u, v satisfy the C-R eqn's :

$$u_x = v_y \ , \ u_y = -v_x$$

Then $v_y = 2y + 2$

Integrating w.r.to y, treating x as constant, we have

$$v = \int (2y+2)\,dy + \phi(x), \text{ where } \phi \text{ is a function of } x$$

$\therefore v = y^2 + 2y + \varphi(x)$ — ①

If φ is differentiable w.r.to x, diff ① w.r.to x, we get

$$v_x = \varphi'(x) = -2x$$

$$\therefore \varphi'(x) = -2x$$

Integrating w.r.to x, we get

$$\varphi(x) = -x^2 + c \quad , c \text{ is a const of integration.}$$

Hence from ① $\quad v(x,y) = y^2 + 2y - x^2 + c$

Hence $f(z) = 2yx + 2x + i\,(y^2 + 2y - x^2 + c)$ is analytic in $\mathbb{C}$.

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ii) $\quad u(x,y) = e^x \cos y$

$u_x = e^x \cos y \quad , u_y = -e^x \sin y$

$u_{xx} = e^x \cos y \quad , u_{yx} = -e^x \sin y = u_{xy} \quad , u_{yy} = -e^x \cos y$.

$u_{xx}, u_{xy}, u_{yx}, u_{yy}$ are all continuous and $u_{xx} + u_{yy} = 0 \quad \forall (x,y) \in \mathbb{R}^2$.

So, u is harmonic.

Hence it is possible to construct an analytic function $f = u + iv$.

Since f is analytic, u, v satisfy the C-R equations:

$$u_x = v_y \quad , u_y = -v_x .$$

Then $v_y = e^x \cos y$, we have

Integrating w.r.to y, treating x as constant,

$$v = \int e^x \cos y \, dy + \phi(x), \text{ where } \phi \text{ is a function of } x.$$

$\therefore v = \cancel{e^x \cos}\ e^x \sin y + \varphi(x)$ ——— ①

If φ is differentiable w.r. to x, diff ① w.r. to x, we get

$$v_x = e^x \sin y + \varphi'(x) = e^x \sin y \quad (\text{by C-R eq}^n)$$

~~Integrating w.r. to x, we get~~

$$v_x = e^x \sin y \quad (\because \varphi'(x) = 0)$$

Integrating w.r. to x, we get

$$v = e^x \sin y + c$$

iii) $u(x,y) = e^x \cos y + y$

$u_x = e^x \cos y \qquad u_y = -e^x \sin y + 1$

$u_{xx} = e^x \cos y \qquad u_{yx} = -e^x \sin y = u_{xy} \quad , u_{yy} = -e^x \cos y$

$u_{xx}, u_{yx}, u_{xy}, u_{yy}$ are all continuous and $u_{xx} + u_{yy} = 0$

$\forall (x,y) \in \mathbb{R}^2$.

So u is harmonic.

Hence it is possible to a construct an analytic function $f = u + iv$

Since f is analytic, u, v satisfy the C-R equations:

$$u_x = v_y \ , \ u_y = -v_x$$

Then $v_y = e^x \cos y$

Integrating w.r. to y, treating x as a constant, we have

$v = e^x \sin y + \varphi(x)$, φ is a function of x.

——— ①

If φ is differentiable w.r.to x, diff ① w.r.to x, we get

$$v_x = e^x \sin y + \varphi'(x)$$

$$= e^x \sin y - 1$$

Now $v_x = -u_y = e^x \sin y - 1$

$\therefore \varphi'(x) = -1$

$$\varphi(x) = -x + C.$$

$$v = e^x \sin y - x + C.$$

---

iv) $u(x,y) = -x^3 + 3xy^2 + 2y + 1$

$u_x = -3x^2 + 3y^2$ $\quad u_y = 6xy + 2$

$u_{xx} = -6x +$ $\quad u_{yy} = 6x$

$u_{yx} = 6y$ $\quad u_{xy} = 6y$

So $u_{xx} + u_{yy} = 0$

$u_{xx}, u_{yx}, u_{xy}, u_{yy}$ are all continuous and $u_{xx} + u_{yy} = 0 \ \forall (x,y) \in \mathbb{R}^2$.

So u is harmonic

Hence it is possible to construct an analytic function $f = u + iv$.

Since f is analytic, u, v satisfy C-R eqns.

$$u_x = v_y, \quad u_y = -v_x$$

Then $v_y = -3x^2 + 3y^2$

Integrating w.r.to y, treating x as a const, we have

$$v = -3x^2 y + y^3 + \varphi(x), \ \varphi \text{ is function of } x.$$

If $\phi(x)$ is differentiable w.r.to x, we get

$$v_x = -6xy + \phi'(x)$$

Now $v_x = -u_y = -6xy - 2$

$$\therefore \phi'(x) = -2$$

$$\phi(x) = -2x + C$$

$$\therefore \quad v = -3x^2y + y^3 - 2x + C.$$