Polynomial Ring

Let $(R,+,\cdot)$ be a ring and $R[X]$ be the collection of polynomials that is

$$R[X]=\left\{f=\sum_{i=0}^{\infty} a_i x^i : a_i \in R \text{ with almost all } a_i=0\right\}.$$

The $(R[X],+,\cdot)$ is also be a ring. This ring is called polynomial ring or polynomial over a ring.

Note: Only the coefficients of the polynomials are the members of ring R.

Polynomial Ring over Commutative Ring

Let $(R,+,\cdot)$ be a commutative ring. Then the polynomial ring $(R[X],+,\cdot)$ is called a polynomial ring over commutative ring.

Remarks

- If R be a commutative ring with unity, then $R[X]$ will also be a commutative ring with unity.

- If two different polynomials define the same function on R, they will be treated as different members of $R[X]$.

Example: Let $f(x) = x \in \mathbb{Z}_3$ and $g(x) = x^3 \in \mathbb{Z}_3$. Here $f(x)$ and $g(x)$ both are defined on ring $\mathbb{Z}_3$, but they are two different polynomials.

$$f(0) = g(0) = 0$$
$$f(1) = g(1) = 1$$
$$f(2) = g(2) = 2$$

Clearly, $f(x)$ and $g(x)$ are two different polynomials but they define the same function on $\mathbb{Z}_3$.

Sum and Product of Polynomial Rings

Suppose $p(x) = \sum_{i=0}^{\infty} a_i x^i \in R[X]$ and $q(x) = \sum_{i=0}^{\infty} b_i x^i \in R[X]$.

Then the sum and product of these two polynomials are

$$\text{sum} = p(x) + q(x) = \sum_{i=0}^{\infty} (a_i + b_i) x^i$$

and $$\text{product} = p(x) \cdot q(x) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} (a_j \cdot b_{i-j}) \right) x^i$$

where $(a_i + b_i)$ and $a_j \cdot b_{i-j}$ are determined using the operations stated on R.

Example: Find the sum and product of the following polynomials
$p(x) = 1 + 4x - 5x^2$ and $q(x) = 2 + 5x - 9x^2$ over the ring (i) $(\mathbb{Z}, +, \cdot)$ and (ii) $(\mathbb{Z}_4, +, \cdot)$ where notations and symbols are their usual meaning.

Ans. Over ring $(\mathbb{Z}, +, \cdot)$

$p(x) = 1 + 4x - 5x^2$. Here $a_0 = 1$, $a_1 = 4$, $a_2 = -5$

$q(x) = 2 + 5x - 9x^2$, Here $b_0 = 2$, $b_1 = 5$, $b_2 = -9$

(i.) Over the ring $(\mathbb{Z}, +, \cdot)$

$$\text{sum} = p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$$
$$= 3 + 9x - 14x^2$$

$\text{Product} = f(x).g(x) = c_0 + c_1 x + c_2 x^2$

where $c_0 = a_0 b_0 = 1.2 = 2$

$c_1 = a_0 b_1 + b_0 a_1 = 5 + 8 = 13$

$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0 = -9 + 20 - 10 = 1$

$\therefore \text{Product} = c_0 + c_1 x + c_2 x^2 = 2 + 13x + x^2$

(ii) <u>over the ring $(Z_4, +, \cdot)$</u>

- $\text{Sum} = f(x) + g(x) = c_0 + c_1 x + c_2 x^2$

where $c_0 = (a_0 + b_0) \pmod 4 = 3 \pmod 4 = 3$

$c_1 = (a_1 + b_1) \pmod 4 = 9 \pmod 4 = 1$

$c_2 = (a_2 + b_2) \pmod 4 = -14 \pmod 4 = 2$

$\therefore \text{sum} = c_0 + c_1 x + c_2 x^2 = 3 + x + 2x^2$

$\text{Product} = p(x).g(x) = d_0 + d_1 x + d_2 x^2$

where $d_0 = (a_0 b_0) \pmod 4 = 2 \pmod 4 = 2$

$d_1 = (a_0 b_1 + b_0 a_1) \pmod 4 = 13 \pmod 4 = 1$

$d_2 = (a_0 b_2 + a_1 b_1 + a_2 b_0) \pmod 4 = 1 \pmod 4 = 1$

$\therefore \text{Product} = d_0 + d_1 x + d_2 x^2 = 2 + x + x^2.$

<u>Assignment</u>

Find the sum and product of the following polynomials

1. $p(x) = 1 + 6x - 7x^2$, $q(x) = 3 + 5x - 7x^2$ over ring (i) $(Z, +, \cdot)$ and (ii) $(Z_3, +, \cdot)$

2. $p(x) = 2 + 3x - 5x^2$, $q(x) = 4 + 2x + 3x^2$ over ring (i) $(Z, +, \cdot)$ and (ii) $(Z_4, +, \cdot)$

3. $p(x) = 3 + x - 4x^2$, $q(x) = 1 + 7x - 5x^2$ over ring (i) $(Z, +, \cdot)$ and (ii) $(Z_5, +, \cdot)$.