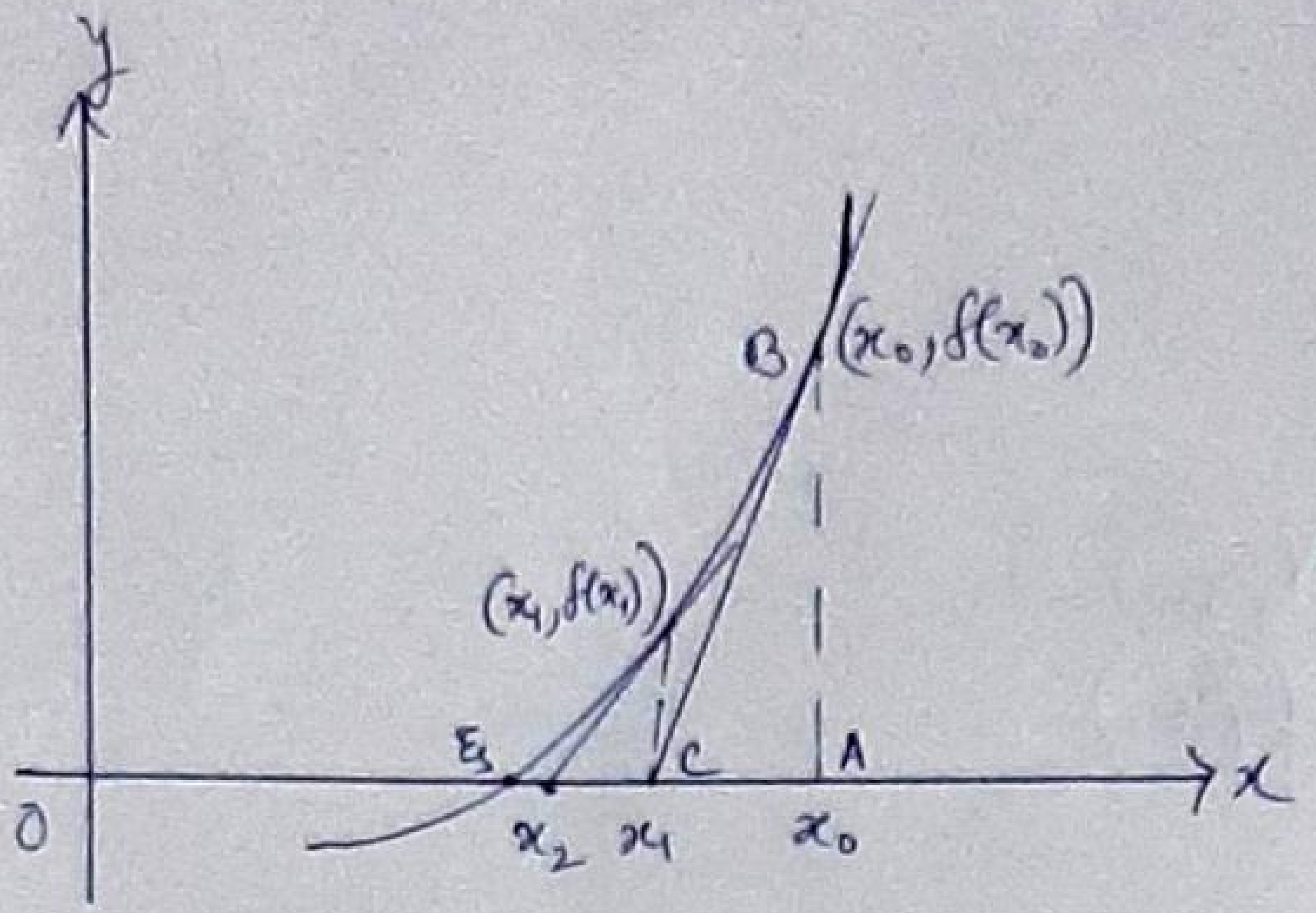
Newton-Raphson Method | Method of Tangents



Geometrical interpretation

In Newton-Raphson (NR) method, a tangent is drawn at the point $(x_0, f(x_0))$ to the curve $y = f(x)$. This tangent meets x-axis at the point $(x_1, 0)$. Again, a tangent is drawn at the point $(x_1, f(x_1))$ which meets x-axis at $(x_2, 0)$.

From ΔABC

$$\tan \angle BCA = \frac{AB}{CA}$$

or, $$f'(x_0) = \frac{f(x_0)}{x_0 - x_1}$$

or $$x_0 - x_1 = \frac{f(x_0)}{f'(x_0)}$$

or $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Eq^n of tangent

$$y - f(x_0) = f'(x_0)(x - x_0)$$

$$0 - f(x_0) = f'(x_0)(x_1 - x_0)$$

or $$x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

or $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Derive Newton-Raphson iteration formula

Let x_0 be the initial approximate root of the equation equation $y = f(x) = 0$ and x_1 be the exact root such that $x_1 = x_0 + h$ where h is the correction of the root or error.

Then x_1 must satisfy $f(x) = 0$

$$\therefore \quad f(x_1) = 0$$

or $$f(x_0 + h) = 0$$

Using Taylor's series we get

$$f(x_0) + hf'(x_0) + \frac{h^2}{2} f''(x_0) + \cdots = 0$$

By neglecting second and higher order derivatives the above equation becomes

$$f(x_0) + h f'(x_0) = 0$$

or, $h = -\dfrac{f(x_0)}{f'(x_0)}$

$\therefore \quad x_1 = x_0 + h$

or, $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

In general

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This is the iterative formula of Newton-Raphson method.

[Note: In Newton-Raphson method, $h = -\dfrac{f(x_0)}{f'(x_0)}$. To calculate h, we neglect the second and higher power of h. So, the value of h calculated by the above formula is not exact. Therefore, the value of x_1 obtain from the formula $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$ is not a root of the equation, but it is a better approximation of x than the approximate root x_0.]

Why Newton-Raphson Method is called method of tangents?

In Newton-Raphson method, a number of tangents are drawn to the curve $y = f(x)$ to get better approximate roots. Therefore, this method is called method of tangents.

Working Rule

Given $f(x) = 0$

Step 1. Find initial root x_0 such that $f(x_0) \simeq 0$, where x_0 is near to root of $f(x) = 0$

Step 2. Calculate $f(x_0)$ and $f'(x_0)$.

Step 3. The the first approximate root by Newton-Raphson method as follows.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Step 4. Calculate $f(x_1)$ and $f(x_1)$. The 2nd approximate root is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Repeat the process untill two consecutive approximations are same upto desired accurancy.

Ex. Find the real root of the equation $x^3 - 3x - 5 = 0$ using Newton-Raphson method correct upto 4 decimal places.

Ans $f(x) = x^3 - 3x - 5 = 0$

$f(2) = -3 < 0$, $f(3) = 13 > 0$

Let $x_0 = 2$

Newton-Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 - 3x_n - 5}{3(x_n^2 - 1)}$$

Put $n = 0$, $x_1 = x_0 - \frac{x_0^3 - 3x_0 - 5}{3(x_0^2 - 1)} = 2 - \frac{(-3)}{9} = 2.3333$

$n = 1$, $x_2 = x_1 - \frac{x_1^3 - 3x_1 - 5}{3(x_1^2 - 1)} = (2.3333) - \frac{(2.3333)^3 - 3(2.3333) - 5}{3\{(2.3333)^2 - 1\}} = 2.2805$

Put $n=2$, $x_3 = x_2 - \dfrac{x_2^3 - 3x_2 - 5}{3(x_2^2-1)} = 2.2790$

$n=3$, $x_4 = x_3 - \dfrac{x_3^3 - 3x_2 - 5}{3(x_3^2-1)} = 2.2790$

Hence, the required root is $x = 2.2790$ correct upto 4 decimal places.

or

Ans $f(x) = x^3 - 3x - 5$

$f(2) = -3 < 0$, $f(3) = 13 > 0$

Let $x_0 = 2$ be the initial approximate root.

The Newton-Raphson iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 - 3x_n - 5}{3(x_n^2-1)} = \frac{2x_n^3 + 5}{3(x_n^2 - 1)}$$

The iteration table is

n	x_n	x_{n+1}
0	2	2.3333
1	2.3333	2.2805
2	2.2805	2.2790
3	2.2790	2.2790

Hence, the required root is $x = 2.2790$ correct upto 4 decimal places.

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