L Hospital's rule (Continuation)

1. Find a, b in order that

$$\lim_{x \to 0} \frac{x(1+a\cos x) - b\sin x}{x^3} = 1.$$

Sol^n :

$$\lim_{x \to 0} \frac{x(1+a\cos x) - b\sin x}{x^3} \quad \left(= \left(\tfrac{0}{0}\right)\right)$$

$$= \lim_{x \to 0} \frac{(1+a\cos x) + x(-a\sin x) - b\cos x}{3x^2}$$

As the denominator vanishes at $x = 0$

$\therefore$ in order to have a finite limit numerator also vanishes at $x = 0$ i.e.,

$$1 + a - b = 0 \quad \text{———} \quad (1)$$

For value a, b satisfying (1) the limit becomes $(0/0)$ form.

$\therefore$ required limit $= \lim_{x \to 0} \dfrac{-2a\sin x + x(-a\cos x) + b\sin x}{6x} \quad \left(= \left(\tfrac{0}{0}\right)\right)$

$$= \lim_{x \to 0} \frac{-2a\cos x - a\cos x + ax\sin x + b\cos x}{6}$$

$$= \frac{1}{6}(-2a - a + b) = 1 \text{ (given)}$$

$$\therefore -3a + b = 6 \quad \text{———} \quad (2)$$

(1) and (2) $\Rightarrow$ $a = -\frac{5}{2}$, $b = -\frac{3}{2}$

2. If $\lim_{x \to 0} \dfrac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$, then find the value of a, b, c.

Sol^n : As the limits exists finitely and denominator vanishes at $x = 0$

$\therefore$ the numerator must also vanishes at $x = 0$ i.e,

$$a - b + c = 0 \quad \text{———} \quad (1)$$

For values of a, b, c satisfying (1) the limit becomes $0/0$ form.

$\therefore$ required limit $= \lim_{x \to 0} \frac{a e^{x} - b \cos x + c e^{-x}}{x \sin x} \left[= \frac{0}{0} \right]$

$$= \lim_{x \to 0} \frac{a e^{x} + b \sin x - c e^{-x}}{x \cos x + \sin x}$$

By the same logic as above

$$a - c = 0 \quad \text{———} \quad (2)$$

and the required limit

$$= \lim_{x \to 0} \frac{a e^{x} + b \sin x - c e^{-x}}{x \cos x + \sin x} \left(= \frac{0}{0} \right)$$

$$= \lim_{x \to 0} \frac{a e^{x} + b \cos x + c e^{-x}}{\cos x - x \sin x + \cos x} = \frac{a+b+c}{2} = 2 \text{ (given)}$$

$$\therefore \quad a + b + c = 4 \quad \text{———} \quad (3)$$

Solving (1), (2) and (3) : $a = 1,\ b = 2,\ c = 1$.

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(3) If $\lim_{x \to 0} \frac{\sin 2x + a \sin x}{x^3}$ be finite, find the value of a and the limit.

Solⁿ: $\lim_{x \to 0} \frac{\sin 2x + a \sin x}{x^3} \left(= \frac{0}{0} \right)$

$$= \lim_{x \to 0} \frac{2\cos 2x + a \cos x}{3x^2}$$

As $x \to 0$ the denominator $3x^2 = 0$ and so for the limit to be finite the numerator must be 0 as $x \to 0$

$$\therefore \quad 2 + a = 0 \quad \text{or} \quad a = -2$$

For $a = -2$ the given limit becomes

$$\lim_{x\to 0} \frac{\sin 2x - 2\sin x}{x^3} \quad \left(= \frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{2\cos 2x - 2\cos x}{3x^2} \quad \left(= \frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{-4\sin 2x + 2\sin x}{6x} \quad \left(= \frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{-8\cos 2x + 2\cos x}{6} = -\frac{6}{6} = -1.$$

(4) If α, β be the roots of $ax^2+bx+c=0$ show that $\lim_{x\to\alpha} \dfrac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2} = \dfrac{1}{2}a^2(\alpha-\beta)^2$

Solⁿ: As α, β be the roots of $ax^2+bx+c=0$.

$\therefore$ $a\alpha^2+b\alpha+c=0$ and $a\beta^2+b\beta+c=0$

Now,

$$\lim_{x\to\alpha} \frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2} \quad \left(= \frac{0}{0}\right)$$

$$= \lim_{x\to\alpha} \frac{(2ax+b)\sin(ax^2+bx+c)}{2(x-\alpha)} \quad \left(= \frac{0}{0}\right)$$

$$= \lim_{x\to\alpha} \frac{2a\sin(ax^2+bx+c) + (2ax+b)^2\cos(ax^2+bx+c)}{2}$$

$$= \frac{1}{2}\left\{2a\sin(a\alpha^2+b\alpha+c) + (2a\alpha+b)^2\cos(a\alpha^2+b\alpha+c)\right\}$$

$$= \frac{1}{2}(2a\alpha+b)^2 = \frac{1}{2}a^2\left(2\alpha+\frac{b}{a}\right)^2$$

$$= \frac{1}{2}a^2(2\alpha-\alpha-\beta)^2$$

$$= \frac{1}{2}a^2(\alpha-\beta)^2$$

EX 1. Find a, b in order that

$$\lim_{x \to 0} \frac{a \sin 2x - b \sin 3x}{5x^3} = 1.$$

(Ans: $a = 3, b = 2$).

2. Find the value of a such that

$$\lim_{x \to 0} \frac{a \sin x - \sin 2x}{\tan^3 x}$$ is finite. Find the limit.

(Ans: $a = 2$, 2)

3. Find a and b such that $\displaystyle\lim_{x \to 0} \frac{a e^{x} + b e^{-x} + 2 \sin x}{\sin x + x \cos x} = 2.$ (Ans: a

$b =$

4. Determine a, b, c such that

$$\lim_{x \to 0} \frac{x(a + b \cos x) + c \sin x}{x^5} = \frac{1}{60}.$$

(Ans: $a = 2, b = 1,$ $c = -3$)

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