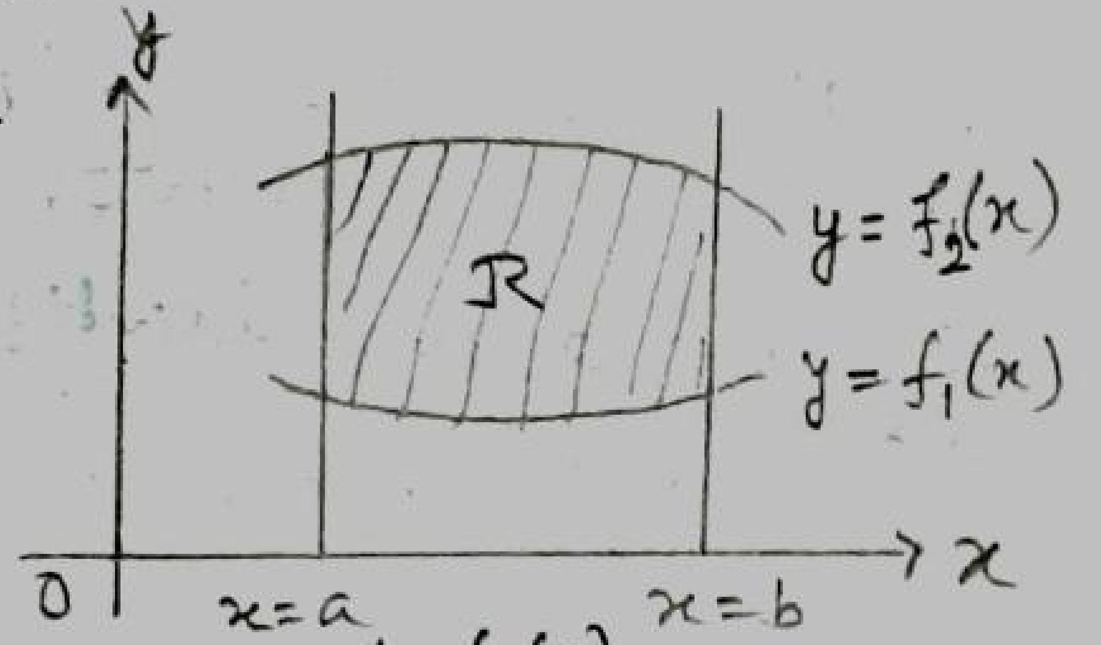
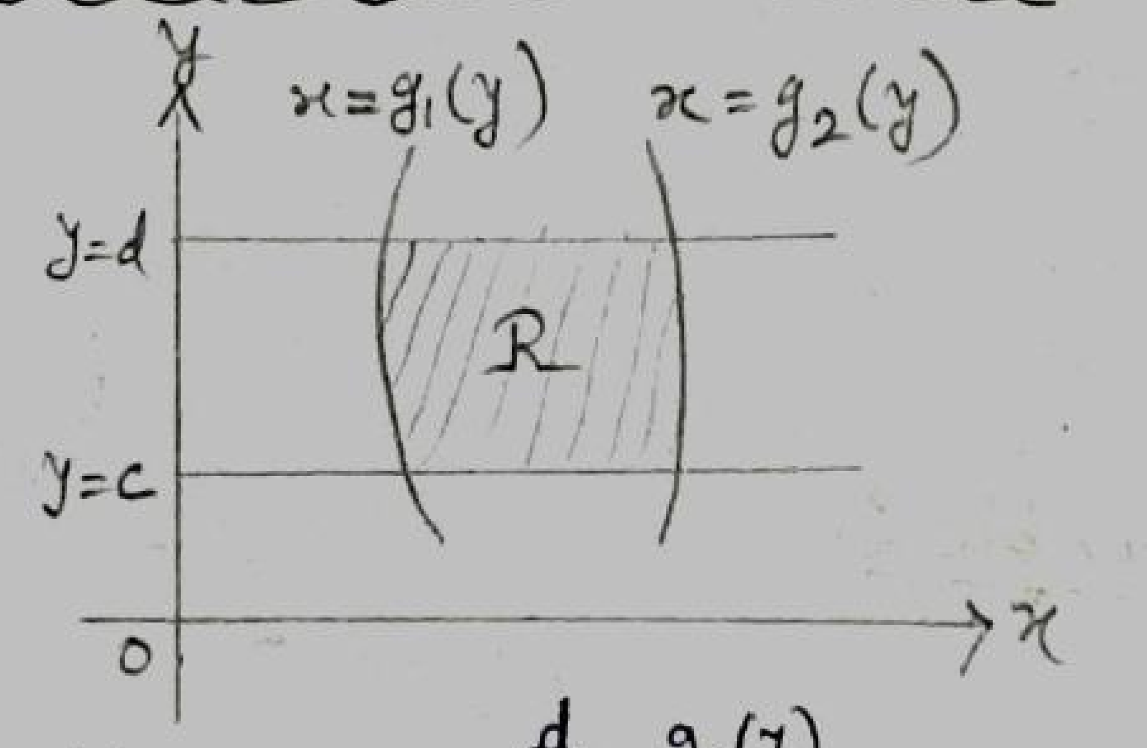
Double Integration over Non-rectangular Region

Case-I



$$\iint_R f(x,y)\,dA = \int_{x=a}^{b}\int_{y=f_1(x)}^{f_2(x)} f(x,y)\,dy\,dx$$

Case II



$$\iint_R f(x,y)\,dA = \int_{y=c}^{d}\int_{x=g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy$$

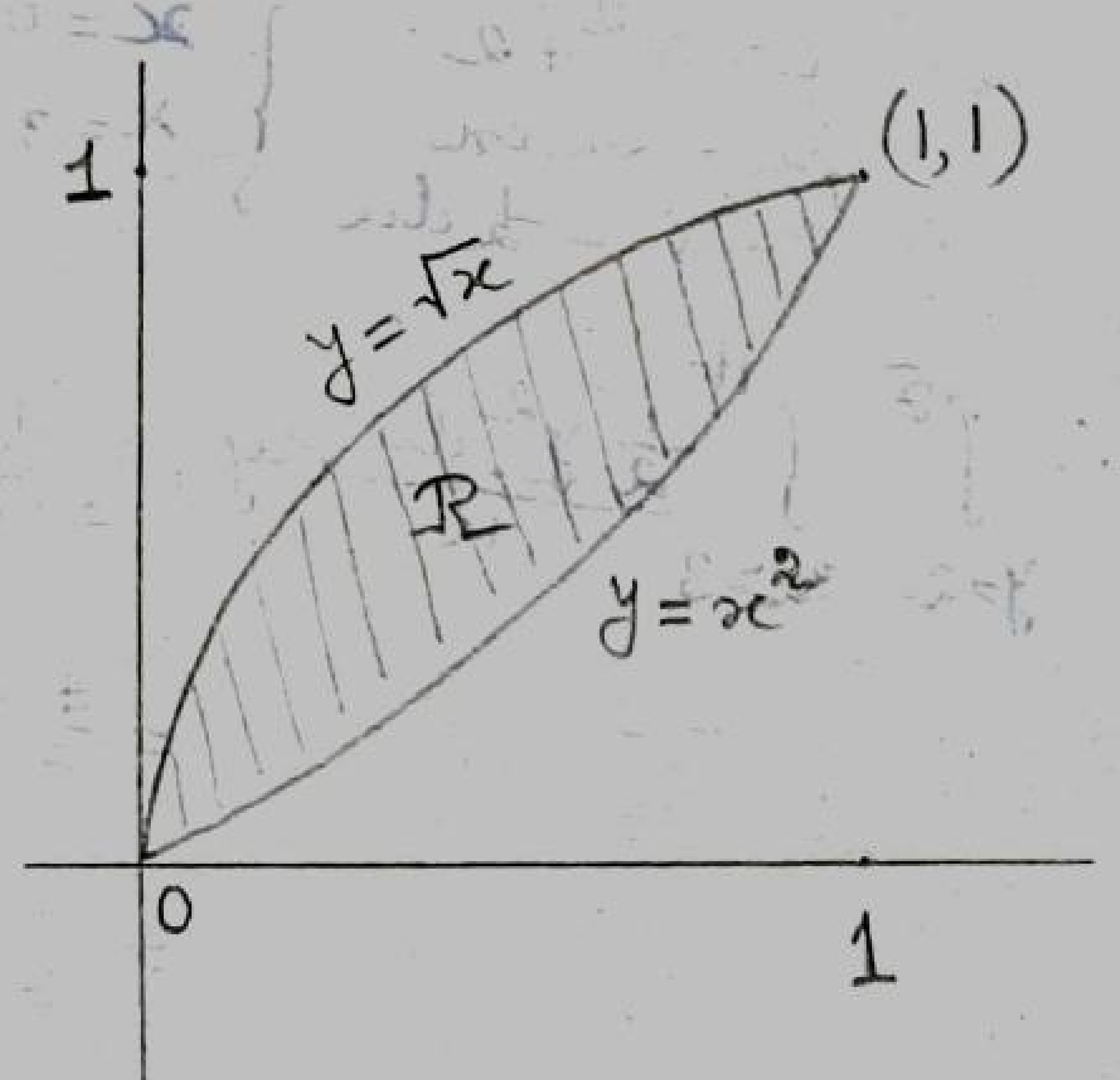
To calculate $\iint_R f(x,y)\,dA$ over non-rectangular region R.

First integrate with respect to that variable whose bounds are functions of the another variable. Then integrate with respect to the variable whose bounds are constants.

In case I, the bounds of y are $y_1 = f_1(x)$ and $y_2 = f_2(x)$ which are variables. So, first integrate with respect to y and then x. In case II, the bounds of x are $x_1 = g_1(y)$ and $x_2 = g_2(y)$ which are variables. So, first integrate with respect to x and then y. [see the following examples].

Example 5. Find the volume of the solid that lies under $f(x,y) = 2x+5y$ and in region R, where R is bounded by $y = x^2$, $y = \sqrt{x}$ and $x = 1$.

Ans.
$$\int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} (2x+5y)\,dy\,dx$$



$$= \int_{x=0}^{1}\left[2xy + \frac{5y^2}{2}\right]_{x^2}^{\sqrt{x}} dx$$

$$= \int_{x=0}^{1}\left(2x^{3/2} + \frac{5x}{2} - 2x^3 - \frac{5}{2}x^4\right)dx$$

$$= \left[\frac{4x^{5/2}}{5} + \frac{5x^2}{4} - \frac{2x^4}{4} - \frac{5}{2}\cdot\frac{x^5}{5}\right]_0^1$$

$$= \left[\frac{4}{5}x^{5/2} + \frac{5}{4}x^2 - \frac{x^4}{2} - \frac{x^5}{2}\right]_0^1$$

$$= \frac{4}{5} + \frac{5}{4} - 1$$

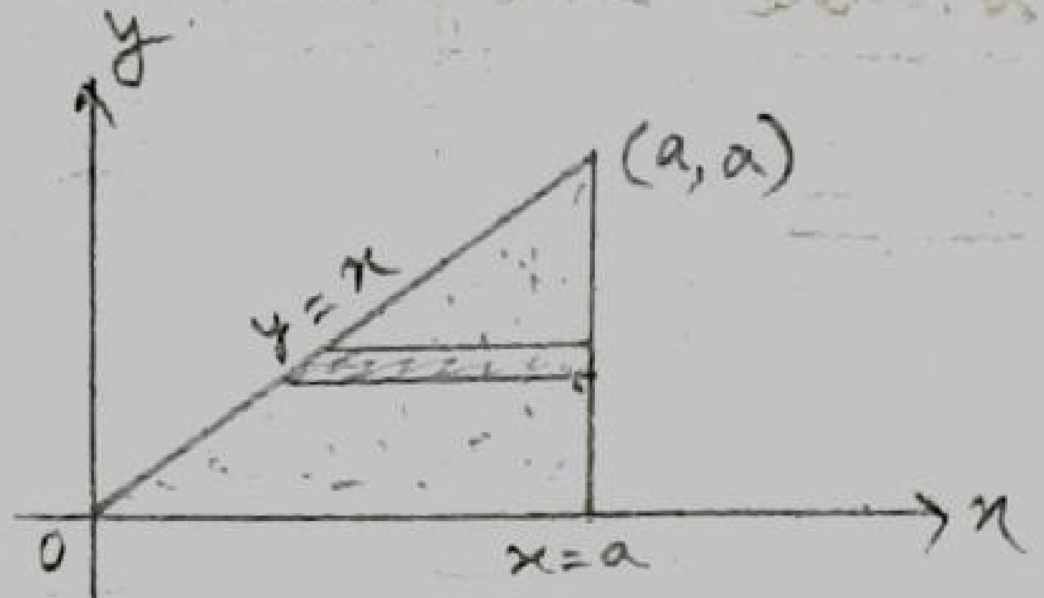
$$= \frac{16+25-20}{20}$$

$$= \frac{21}{20}$$

$$= 1\frac{1}{20}$$

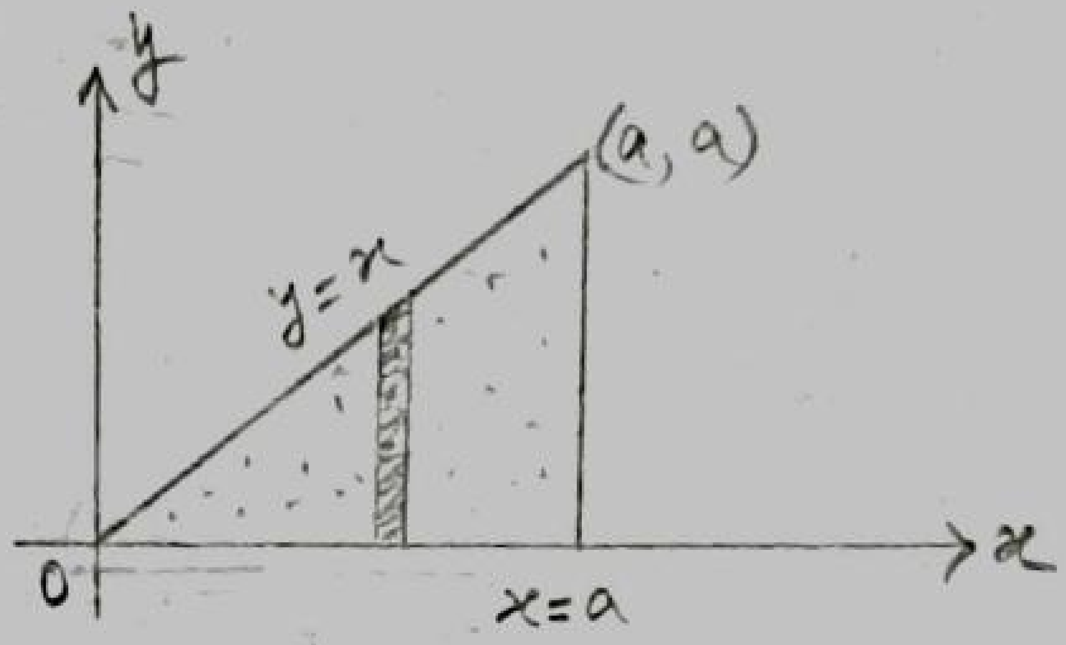
Change order of integration

$$\int_{y=0}^{a}\int_{x=y}^{a} f(x,y)\,dx\,dy$$



By changing the order of integration

$$\int_{x=0}^{a}\int_{y=0}^{x} f(x,y)\,dy\,dx$$

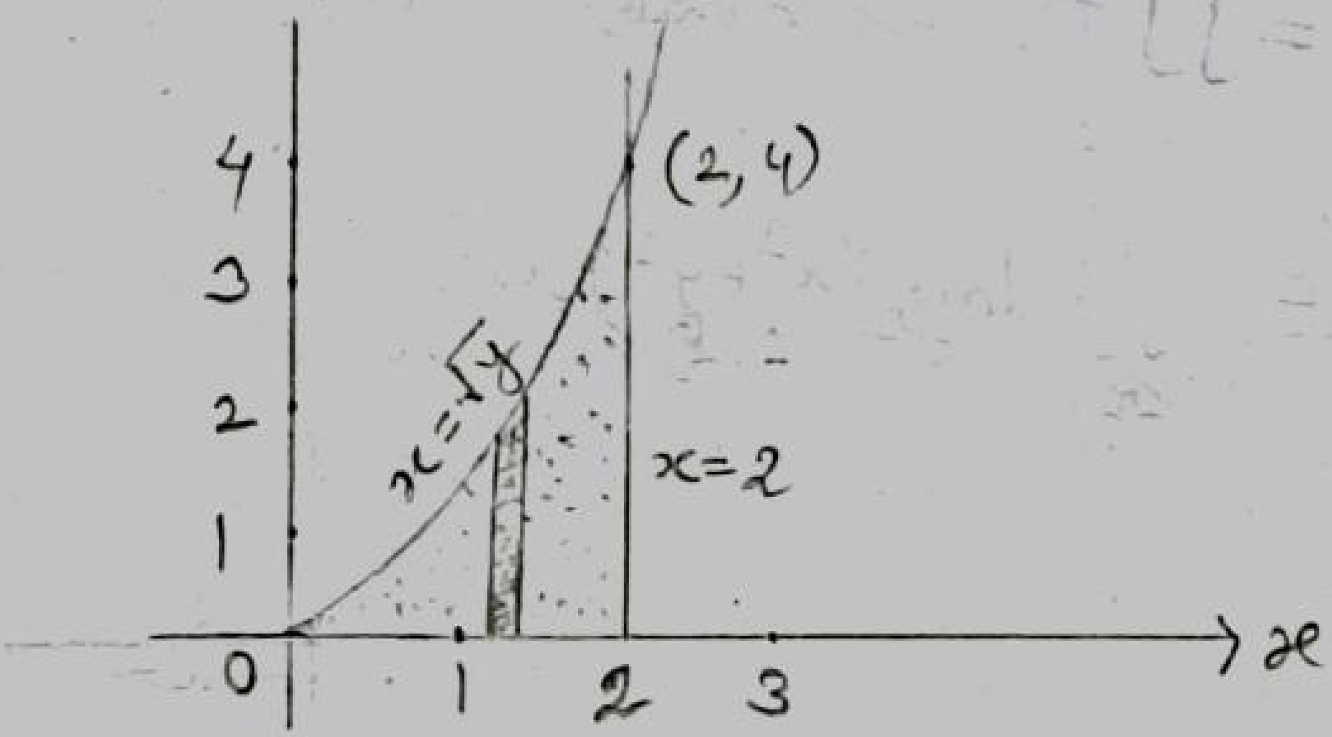


Example 6. $\displaystyle\int_{y=0}^{4}\int_{x=\sqrt{y}}^{2} x(x^2+y^2)\,dx\,dy$. (i) Draw the region of integration.

(ii) Change the order of integration.

(iii) Evaluate the integration you found in (ii).

Ans.



$x=\sqrt{y}$

$y = x^2$

$= 2^2$

$= 4$

$$\int_{y=0}^{4}\int_{x=\sqrt{y}}^{2} x(x^2+y^2)\,dx\,dy = \int_{x=0}^{2}\int_{y=0}^{x^2} x(x^2+y^2)\,dy\,dx$$

$$= \int_{x=0}^{2}\left[x^3y + \frac{xy^3}{3}\right]_0^{x^2} dx$$

$$= \int_{x=0}^{2}\left(x^5 + \frac{x^7}{3}\right)dx$$

$$= \left[\frac{x^6}{6} + \frac{x^8}{24}\right]_0^2$$

$$= \frac{\cancel{64}^{\,32}}{\cancel{6}_{\,3}} + \frac{\cancel{256}^{\,32}}{\cancel{24}}$$

$$= \frac{32}{3} + \frac{32}{3}$$

$$= \frac{64}{3}$$

$$= 21\frac{1}{3}$$

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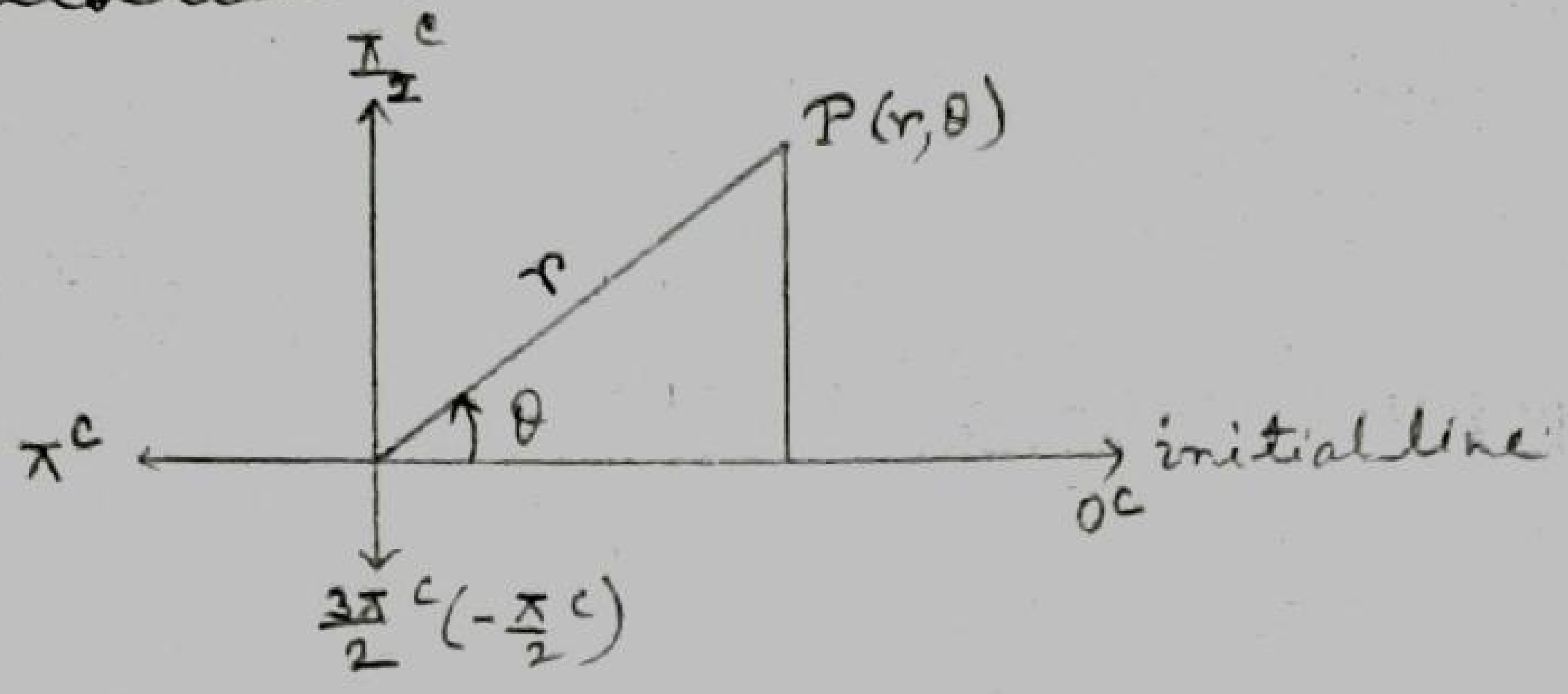
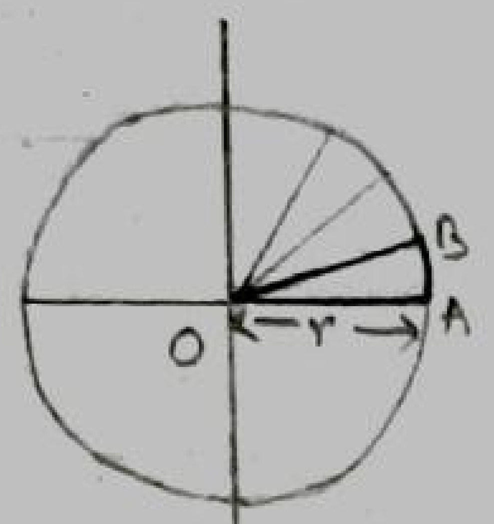
Double Integration in Polar Coordinate

Case - I



Case - II



Changin double integration from cartesian to polar co-ordinate.

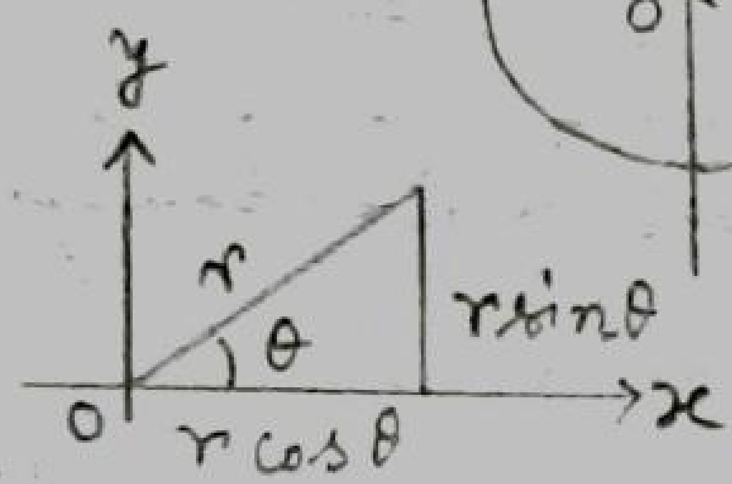
$$\iint_R f(x,y)\,dx\,dy = \iint_S f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$$

where R is the region in xy-plane and S is the region in $r\theta$-plane.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r$$



$$\left.\begin{aligned} x &= r\cos\theta, \ \frac{\partial x}{\partial r} = \cos\theta, \ \frac{\partial x}{\partial \theta} = -r\sin\theta \\ y &= r\sin\theta, \ \frac{\partial y}{\partial r} = \sin\theta, \ \frac{\partial y}{\partial \theta} = r\cos\theta \end{aligned}\right\}$$

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Example 7. Evaluate $\iint_R \log_e(x^2+y^2)\,dx\,dy$, where R: inside $x^2+y^2=a^2$

Ans. Let $x = r\cos\theta$

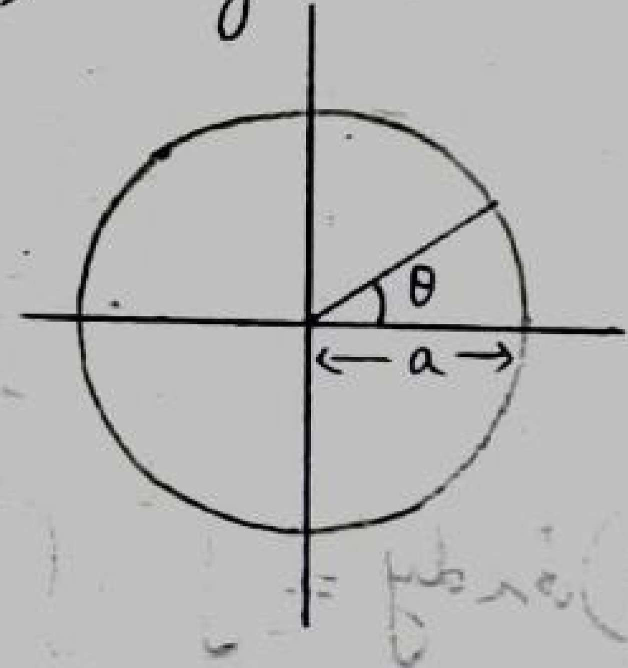
$y = r\sin\theta$

$x^2+y^2 = r^2 = a^2$

$r = \pm a$

$0 \leq r \leq a$

$0 \leq \theta \leq 2\pi$



$$\iint_R \log_e(x^2+y^2)\,dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=0}^{a} \log_e(r^2)\, r\,dr\,d\theta = 2\int_{\theta=0}^{2\pi}\int_{r=0}^{a} \log r \cdot r\,dr\,d\theta$$

$$= 2\int_{\theta=0}^{2\pi}\left[\frac{r^2}{2}\log r - \int \frac{1}{r}\cdot\frac{r^2}{2}\,dr\right]_0^a d\theta$$

$$= 2\int_0^{2\pi}\left[\frac{r^2}{2}\log r - \frac{r^2}{4}\right]_0^a d\theta$$

$$= 2\left(\frac{a^2}{2}\log a - \frac{a^2}{4}\right)\int_0^{2\pi} d\theta$$

$$= \frac{\cancel{2}a^2}{\cancel{2}}\left(\log a - \frac{1}{2}\right)[\theta]_0^{2\pi}$$

$$= \frac{2\pi a^2}{2}(2\log a - 1)$$

$$= \pi a^2(2\log a - 1)$$

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Example 8. Find the double integral $\int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} e^{-(x^2+y^2)}\,dx\,dy$

Ans. $x = \sqrt{a^2-y^2}$

or, $x^2 = a^2-y^2$

or, $x^2+y^2 = a^2$

or, $r^2 = a^2$

or, $r = a$

$0 \le r \le a$

If $x = y$ then

or, $x^2+y^2 = a^2$

or, $y^2+y^2 = a^2$

or, $2y^2 = a^2$

or, $y^2 = \frac{a^2}{2}$

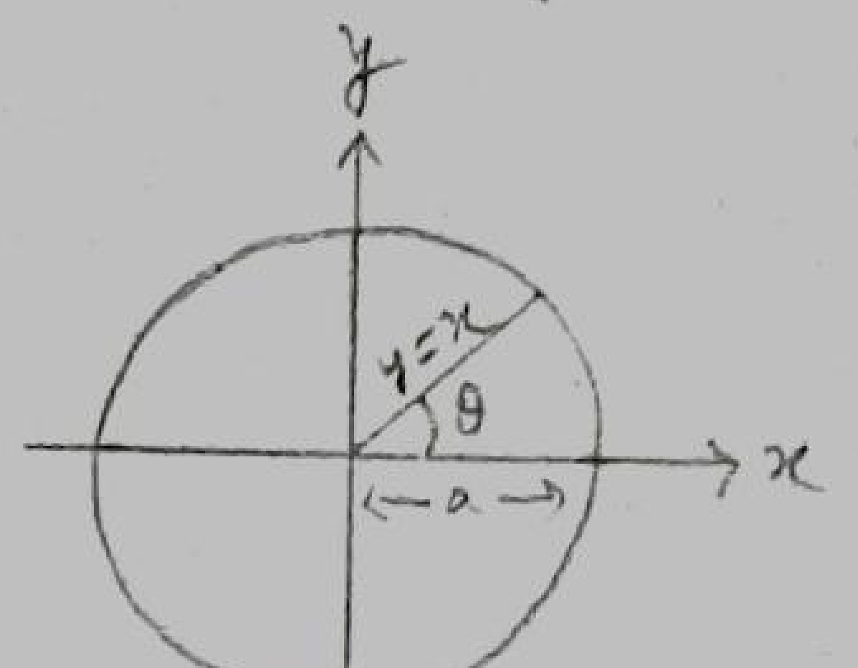
or $y = \sqrt{\frac{a^2}{2}} = \frac{a}{\sqrt{2}}$

$a\sin\theta = \frac{a}{\sqrt{2}}$

$\sin\theta = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$

$\theta = \frac{\pi}{4}$

$0 \le \theta \le \frac{\pi}{4}$



$$\int_{\theta=0}^{\pi/4}\int_{r=0}^{a} e^{-r^2}\cdot r\,dr\,d\theta = \int_{\theta=0}^{\frac{\pi}{4}} \left[-\frac{1}{2}e^{-r^2}\right]_0^a d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{4}} -\frac{1}{2}\left(e^{-a^2}-1\right)d\theta$$

$$= -\frac{1}{2}\left(e^{-a^2}-1\right)[\theta]_0^{\pi/4}$$

$$= \frac{\pi}{8}\left(1-e^{-a^2}\right).$$

Let $r^2 = u$

$2r\,dr = du$

$r\,dr = \frac{1}{2}du$

$\int e^{-r^2}\cdot r\,dr = \frac{1}{2}\int e^{-u}\,du$

$= -\frac{1}{2}e^{-u} = -\frac{1}{2}e^{-r^2}$

Assignments

1. Find the volume of the solid that lies under
   (i) $f(x,y) = x^2+y^2$ and in region R bounded by $y=2x$, $y=x^2$.
   (ii) $z = xy$ and in region R bounded by $y=x-1$; $y^2=2x+6$.

2. Change the order of integration and evaluate them.
   (i) $\int_0^1\int_x^1 x\sqrt{1+y^3}\,dy\,dx$ (iii) $\int_0^3\int_2^9 \sqrt{x}\cos x\,dx\,dy$
   (ii) $\int_0^4\int_{x^2/4}^{2\sqrt{x}} dy\,dx$ (iv) $\int_0^a\int_0^x \frac{dy\,dx}{(y+a)\sqrt{(a-x)(x-y)}}$

3. Express the following integral in polar coordinate and evaluate them.
   (i) $\int_0^2\int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$ (iii) $\int_0^a\int_y^a \frac{x^2}{(x^2+y^2)^{3/2}}\,dx\,dy$
   (ii) $\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\,dy\,dx$