Differentiation

Let f be a real valued function defined on an interval $I = [a,b] \subseteq R$.

1) Let c be an interior point of I i.e $a < c < b$.

f is said to be differentiable or derivable at c if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \quad \text{~~exists f~~ or} \quad \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \quad \text{exists finitely.}$$

If the limit be l, l is called the derivative or the differential coefficient of the function f at $x = c$, and is denoted by $f'(c)$ or $\left.\frac{df}{dx}\right|_{\text{at } x=c}$

The limit ~~in case it~~ exists, when the left-hand and right-hand limit exist ~~are~~ and are equal.

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \quad \text{or} \quad \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$$ is called Left-hand derivative and is denoted by $f'(c-0)$ or $Lf'(c)$.

while $$\lim_{x \to c^+} \frac{f(x) + f(c)}{x - c} \quad \text{or} \quad \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$$ is called Right-hand derivative and is denoted by $f'(c+0)$ or $Rf'(c)$.

Thus, the derivative $f'(c)$ exist when $Lf'(c) = Rf'(c)$.

2) f is said to be derivable at the left end point a if

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$$ exists finitely, and is denoted by $f'(a)$.

3) f is said to be derivable at the right end point b if

$$\lim_{x \to b^-} \frac{f(x) - f(b)}{x - b}$$ exists finitely and is denoted by $f'(b)$.

Note 1. If $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ is either $+\infty$ or $-\infty$, we say that infinite derivative exists at c and $f'(c) = \infty$ or $-\infty$ respectively. By the phrase 'derivative at c' we will mean that $f'(c)$ is finite.

2. Let $D \subset R$ and $f: D \to R$. It is possible to differentiability of f at a point $c \in D$, provided $c \in D \cap D'$.

3. If f be differentiable at every pt of D, it is said to be differentiable on D.

Example

A function f defined ~~by~~ on R by

$$f(x) = x \quad \text{if } 0 \le x < 1$$
$$= 1 \quad \text{if } x \geq 1.$$

Show that f is not differentiable at $x = 1$.

Ans

$$\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{x-1}{x-1} = 1 \Rightarrow Lf'(1) = 1$$

$$\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{1-1}{x-1} = 0 \Rightarrow Rf'(1) = 0$$

$$Lf'(1) \neq Rf'(1)$$

Thus $\lim_{x\to 1} \frac{f(x)-f(1)}{x-1}$ does not exist i.e., $f'(1)$ does not exist.

Example Show that the function $f(x) = x^2$ is derivable on $[0,1]$

Ans Let c be any point in $(0,1)$, then

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c} \frac{x^2-c^2}{x-c} = \lim_{x\to c} (x+c)$$
$$= 2c.$$

At the end points, we have

$$\lim_{x\to 0+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0+} \frac{x^2}{x} = \lim_{x\to 0+} x = 0 \Rightarrow f'(0) = 0$$

$$\lim_{x\to 1-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1-} \frac{x^2-1}{x-1} = \lim_{x\to 1-} (x+1) = 2 \Rightarrow f'(1) = 2$$

Thus f is derivable on $[0,1]$

Relation between Continuity and differentiability

Th A function which is derivable at a point is necessarily Continuous at that point (derivability at a point $\Rightarrow$ Continuity at that point)

Proof Let $f: I \to R$ be ~~Continuous~~ differentiable at $c \in I$.

$\therefore$ $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists finitely and equal to $f'(c)$

Now $\lim_{x\to c} [f(x)-f(c)] = \lim_{x\to c} \left(\frac{f(x)-f(c)}{x-c}\right)(x-c)$

$$= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \cdot \lim_{x\to c} (x-c)$$

$$= f'(c)\cdot 0 = 0$$

$$\Rightarrow \lim_{x\to c} f(x) = f(c)$$

So f is Continuous at $x = c$

The continuity of f at ~~at~~ point $c\in I$ does not ensure differentiablity of f at c.

For example, let $f(x) = |x|$, $x \in R$

~~At $x=0$, $f(x)=0$. Also f is Contin.~~

$$f(x) = \begin{cases} x & \text{when } x > 0 \\ 0 & \text{when } x = 0 \\ -x & \text{when } x < 0 \end{cases}$$

Since $f(0+0) = 0$, $f(0-0) = 0$ and $f(0) = 0$, so

f is Continuous at 0

But $\displaystyle\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x - 0}{x - 0} = 1 \Rightarrow Rf'(0) = 1.$

$\displaystyle\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{-x - 0}{x - 0} = -1 \Rightarrow Lf'(0) = -1$

$$Lf'(0) \neq Rf'(0)$$

Hence f is not differentiable at $x = 0$

Example

Let $$f(x) = \begin{cases} x \sin \frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

f is Continuous at $x = 0$

But $\displaystyle\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x \sin \frac{1}{x}}{x}$

$\displaystyle = \lim_{x \to 0} \sin \frac{1}{x}$ does not exist

$\therefore$ f is not derivable at $x = 0$

function f is defined on some nbd $N(0)$ of 0 by

$$f(x) = \frac{x}{1+e^{1/x}}, \quad x \neq 0$$
$$= 0, \quad x = 0$$

Find $Lf'(0)$ and $Rf'(0)$. Show that f is not differentiable at 0.

Ans

$$\lim_{x\to 0+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{\frac{x}{1+e^{1/x}} - 0}{x-0}$$

$$= \lim_{x\to 0+} \frac{1}{1+e^{1/x}} = 0 \quad \left(\because \lim_{x\to 0+} e^{1/x} = \infty\right)$$

$\therefore Rf'(0) = 0$

$$\lim_{x\to 0-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0-} \frac{\frac{x}{1+e^{1/x}} - 0}{x-0}$$

$$= \lim_{x\to 0-} \frac{1}{1+e^{1/x}} = 1 \quad \left(\because \lim_{x\to 0-} e^{1/x} = 0\right)$$

$\therefore Lf'(0) = 1$

Since $Lf'(0) \neq Rf'(0)$, so f is not differentiable at $x=0$

A function $f: \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = |x| + |x-1| + |x-2|, \quad x \in \mathbb{R}$$

Find the derived function f' and specify the domain of f'.

Ans

$$f(x) = \begin{cases} x + (x-1) + (x-2) & \text{if } x \geq 2 \\ x + (x-1) + 2 - x & \text{if } 1 \leq x < 2 \\ x + 1 - x + 2 - x & \text{if } 0 \leq x < 1 \\ -x + 1 - x + 2 - x & \text{if } x < 0 \end{cases}$$

$$\therefore f(x) = \begin{cases} 3x-3 & \text{if } x \geq 2 \\ x+1 & 1 \leq x < 2 \\ 3-x & 0 \leq x < 1 \\ 3-3x & x < 0 \end{cases}$$

$$\lim_{h\to 0+} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0+} \frac{(2+h)3-3-3}{h} = 3$$

$$\lim_{h\to 0-} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0-} \frac{(2+h)+1-3}{h} = 1$$

$\therefore$ f is not derivable at $x=2$.

$$\lim_{h\to 0+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0+} \frac{(1+h)+1-2}{h} = 1$$

$$\lim_{h\to 0-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0-} \frac{3-(1+h)-2}{h} = -1$$

$\therefore$ f is not differentiable at $x=1$.

$$\lim_{h\to 0+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0+} \frac{3-(h)-3}{h} = -1$$

$$\lim_{h\to 0-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0-} \frac{3-3h-3}{h} = -3$$

$\therefore$ f is not differentiable at $x=0$

$\therefore$ when $x>2$ $\quad f'(x) = 3$.

when $1<x<2$ $\quad f'(x) = 1$

when $0<x<1$ $\quad f'(x) = -1$

when $x<0$ $\quad f'(x) = -3$

A function f is defined by

$$f(x) = \frac{x\left(e^{1/x} - e^{-1/x}\right)}{\left(e^{1/x} + e^{-1/x}\right)}, \quad x \neq 0$$

$$= 0, \quad x = 0$$

Show that f is continuous at 0 but not differentiable at 0.

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x\left(e^{1/x} - e^{-1/x}\right)}{\left(e^{1/x} + e^{-1/x}\right)} = \lim_{x \to 0^+} \frac{x\left(1 - e^{-2/x}\right)}{\left(1 + e^{-2/x}\right)}$$

$$= 0$$

$$f(0) = 0$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{x\left(e^{1/x} - e^{-1/x}\right)}{\left(e^{1/x} + e^{-1/x}\right)} = \lim_{x \to 0^-} \frac{x\left(e^{2/x} - 1\right)}{\left(e^{2/x} + 1\right)} = 0$$

$\therefore$ f is continuous at $x = 0$.

Now,

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h\left(e^{1/h} - e^{-1/h}\right)}{h\left(e^{1/h} + e^{-1/h}\right)}$$

$$= \lim_{h \to 0^+} \frac{1 - e^{-2/h}}{1 + e^{-2/h}} = 1$$

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{h\left(e^{1/h} - e^{-1/h}\right)}{h\left(e^{1/h} + e^{-1/h}\right)}$$

$$= \lim_{h \to 0^-} \frac{e^{2/h} - 1}{e^{2/h} + 1} = -1$$

$\therefore$ $Rf'(0) = 1$

$Lf'(0) = -1$

Hence f is not diff at $x = 0$

A function f is defined on $(-1,1)$ by

$$f(x) = x^{\alpha} \sin \frac{1}{x^{\beta}} \quad , x \neq 0$$
$$\quad\;\; = 0 \quad\quad\quad\quad , x = 0$$

Prove that (i) if $0 < \beta < \alpha - 1$, f' is continuous at 0
(ii) if $0 < \alpha - 1 \leq \beta$, f' is discontinuous at 0

Ans

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^{\alpha} \sin \frac{1}{h^{\beta}}}{h}$$

$$= \lim_{h \to 0} h^{\alpha - 1} \sin \frac{1}{h^{\beta}}$$

If $\alpha - 1 > 0$ $\quad \lim_{h \to 0} h^{\alpha-1} \sin \frac{1}{h^{\beta}} = 0$

$\therefore$ If $\alpha - 1 > 0$ $\quad f$ is diff at $x = 0$ and $f'(0) = 0$

when $x \neq 0$, $f'(x) = \alpha x^{\alpha - 1} \sin \frac{1}{x^{\beta}} - \beta x^{\alpha} \left(\cos \frac{1}{x^{\beta}}\right) x^{-\beta - 1}$

$$= \alpha x^{\alpha - 1} \sin \frac{1}{x^{\beta}} - \beta x^{\alpha - \beta - 1} \cdot \cos \frac{1}{x^{\beta}}.$$

If $\alpha - \beta - 1 > 0$ then

$$\lim_{x \to 0} f'(x) = 0$$

If $\alpha - \beta - 1 \leq 0$ then $\lim_{x \to 0} f'(x)$ does not exist.

$\therefore$ i) If $0 < \beta < \alpha - 1$, f' is continuous at 0

ii) If $0 < \alpha - 1 \leq \beta$ then f' is discontinu at 0

$f: R \to R$ is defined by $f(x) = e^{-\frac{1}{x^2}} \sin \frac{1}{x}, \ x \neq 0$

$= 0, \quad x = 0$

Show that f' is continuous at 0.

Ans

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{e^{-1/h^2} \sin \frac{1}{h}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} e^{-1/h^2} \sin \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{1}{h^2} e^{-1/h^2} \cdot h \sin \frac{1}{h}$$

Now $\lim_{h \to 0} h \sin \frac{1}{h} = 0$ as $\sin \frac{1}{h}$ is bounded and $\lim_{h \to 0} h = 0$

$$\lim_{h \to 0} \frac{1}{h^2} e^{-1/h^2} = \lim_{K \to \infty} \frac{K}{e^K} \left(\frac{\infty}{\infty}\right) \left[\text{where } K = \frac{1}{h^2}\right]$$

$$= \lim_{K \to \infty} \frac{1}{e^K} = 0$$

$$\therefore \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = 0. \text{ Hence } f'(0) = 0$$

$$\therefore f'(x) = \begin{cases} \frac{e^{-1/x^2}}{x^3} \left[2 \sin \frac{1}{x} - x \cos \frac{1}{x}\right], & x \neq 0 \\ 0 & , x = 0 \end{cases}$$

Now $\lim_{x \to 0} f'(x) = \lim_{x \to 0} \frac{e^{-1/x^2}}{x^4} \left[2x \sin \frac{1}{x} - x^2 \cos \frac{1}{x}\right] = 0$

as

$$\lim_{x \to 0} \frac{e^{-1/x^2}}{x^4} = \lim_{y \to \infty} \frac{y^2}{e^y} \left(\frac{\infty}{\infty}\right) \left(\text{where } y = \frac{1}{x^2}\right)$$

$$= \lim_{y \to \infty} \left(\frac{2y}{e^y}\right) \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{y \to \infty} \frac{2}{e^y} = 0$$

and $\lim_{x \to 0} 2x \sin \frac{1}{x} = 0 \qquad \lim_{x \to 0} x^2 \cos \frac{1}{x} = 0$

$$\therefore \lim_{x \to 0} f'(x) = 0 = f'(0)$$

So f' is continuous at 0

A function $f: R \to R$ is defined by

$$f(x) = x^2 \text{ if } x \text{ is rational}$$
$$= 0 \text{ if } x \text{ is irrational}$$

Show that f is differentiable at 0 and no other point.

Ans

$\cancel{\lim_{x \to 0} f(x)}$

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \begin{cases} \lim_{h \to 0} \frac{h^2}{h} = 0 & \text{when } h \text{ is rational} \\ \lim_{h \to 0} \frac{0}{h} = 0 & \text{when } h \text{ is irrational} \end{cases}$$

$\therefore$ $f'(0)$ exists and equal to 0.

~~Let c be any ~~other pt~~ pt other than (0).~~

~~Case I. Let c be rational.~~

~~Now~~ $\cancel{\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}}$

The function f is discontinuous at all pts other than 0.

$\left(\because \lim_{x \to c} f(x) \text{ does not exist} \right)$

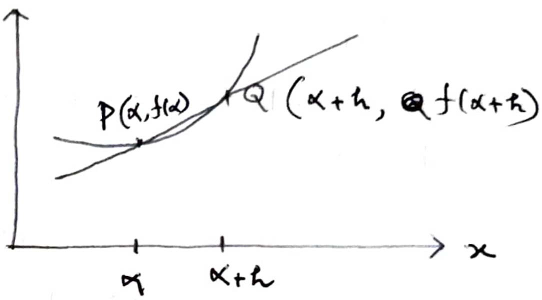
$\therefore$ The function f is not diff at ~~all ther~~ all pts other than 0

Geometrical representation of derivative

Let $y = f(x)$ be the equation of the curve of the function f.

Let $P(\alpha, f(\alpha))$ be ~~any~~ a pt on the curve.

Let $Q(\alpha+h, f(\alpha+h))$ be a neighbouring point of P.



So the gradient of the chord PQ

$$= \frac{f(\alpha+h) - f(\alpha)}{h}$$

when $h \to 0$, Then $Q \to P$.

When $Q \to P$, the chord PQ becomes the tangent at P.

So $\lim\limits_{h \to 0} \frac{f(\alpha+h) - f(\alpha)}{h}$ is the gradient of the tangent at P.

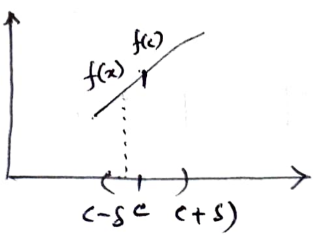
Now $\lim\limits_{h \to 0} \frac{f(\alpha+h) - f(\alpha)}{h} = f'(\alpha)$

$\therefore$ $f'(\alpha)$ is the gradient of the tangent at P.

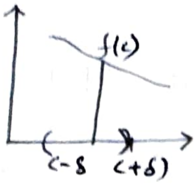
Increasing and decreasing functions

Let $f: I \to R$ be a function.

1. f is said to be increasing at $c \in I$ (where c is an interior pt) if $\exists\ \delta > 0$ s.t $f(x) < f(c)\ \forall\ x \in (c-\delta, c) \cap I$ and $f(x) > f(c)\ \forall\ x \in (c, c+\delta) \cap I$.



2. f is said to be decreasing at $c \in I$ (interior pt of I)
if ~~$f(x) > f(c)$~~ $\exists\ \delta > 0$ s.t



$$f(x) > f(c) \quad \forall\ x \in (c-\delta, c) \cap I$$
and $$f(x) < f(c) \quad \forall\ x \in (c, c+\delta) \cap I.$$

3. f is said to be increasing (or decreasing) at left end pt $c \in I$ if

$$f(x) > f(c) \quad \forall\ x \in (c, c+\delta) \cap I$$
$$\left(\text{or } f(x) < f(c) \quad \forall\ x \in (c, c+\delta) \cap I \text{ respectively.}\right)$$

4. f is said to be increasing (or decreasing) at right end pt $c \in I$ if

$$f(x) < f(c) \quad \forall\ x \in (c-\delta, c) \cap I$$
$$\left(\text{or } f(x) > f(c) \quad \forall\ x \in (c-\delta, c) \cap I\right) \text{ respectively.}$$

Sign of derivative

Th Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ is differentiable at $c \in I$.

i) If $f'(c) > 0$ then f is increasing at c

ii) If $f'(c) < 0$ then f is decreasing at c.

Proof (i) By definition

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

Let us choose $\epsilon = \frac{1}{2}|f'(c)| > 0$

So for this $\epsilon > 0\ \exists\ \delta > 0$ s.t $f'(c) - \epsilon < \dfrac{f(x) - f(c)}{x - c} < f'(c) + \epsilon$

$$\forall\ x \in (c-\delta, c+\delta) \cap I$$

i) ϵ ~~Let~~ If $f'(c) > 0$, then $f'(c) - \epsilon$

$$= f'(c) - \tfrac{1}{2}|f'(c)|$$
$$= f'(c) - \tfrac{1}{2} f'(c)$$
$$= \tfrac{1}{2} f'(c) > 0$$

$$\therefore \frac{f(x) - f(c)}{x - c} > 0 \quad \forall \; x \in (c-\delta, c+\delta) \cap I.$$

When $x \in (c-\delta, c)$, then $x - c < 0$

$$\therefore f(x) - f(c) < 0$$
$$\Rightarrow f(x) < f(c)$$

When $x \in (c, c+\delta)$, then $x - c > 0$

$$\therefore f(x) - f(c) > 0$$
$$\Rightarrow f(x) > f(c)$$

$\therefore$ $f(x) < f(c) \quad \forall \; x \in (c-\delta, c) \cap I.$

& $f(x) > f(c) \quad \forall \; x \in (c, c+\delta) \cap I$

Hence f is increasing at c

ii) If $f'(c) < 0$, then $f'(c) + \epsilon = f'(c) - \tfrac{1}{2} f'(c)$

$$= f'(c) < 0.$$

$$\therefore \frac{f(x) - f(c)}{x - c} < 0$$

$\therefore$ $f(x) > f(c) \quad \forall \; x \in (c-\delta, c) \cap I.$

$f(x) < f(c) \quad \forall \; x \in (c, c+\delta) \cap I$

Hence f is decreasing at c

Note 1. If $f'(c) = 0$, no conclusion can be drawn.

Ex Let $f(x) = x^3$. Then $f'(x) = 3x^2$ and $f'(0) = 0$.

In $0 < x < 0 + \delta$, $f(x) - f(0) > 0$ and

in $0 - \delta < x < 0$, $f(x) - f(0) < 0$

So f is increasing at 0

Ex Let $f(x) = x^2$. Then $f'(x) = 2x$ and $f'(0) = 0$

$f(x) - f(0) > 0$ in both $0 - \delta < x < 0$, $0 < x < 0 + \delta$.

So f is neither increasing nor decreasing at 0.

Note 2. A function f may be increasing (or decreasing) at point c in its domain without being differentiable at c.

Ex Let $f(x) = x$, $x < 1$
$= 2x - 1$, $x \geq 1$.

f is not differentiable at $x = 1$.
But f is increasing at $x = 1$

Ex Let $f(x) = 1 - x$, $x < 0$
$= 1 - 2x$, $x \geq 0$

f is not diff at 0. But f is decreasing at 0.

Details

$$\lim_{x \to 1-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1-} \frac{x - 1}{x - 1} = 1$$

$$\lim_{x \to 1+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1+} \frac{2x - 1 - 1}{x - 1} = 2$$

$\therefore$ f is not diff at $x = 1$.

In $1 - \delta < x < 1$, $f(x) < f(1)$. & in $1 < x < 1 + \delta$, $f(x) > f(1)$.

$\therefore$ f is $\uparrow$ at 1

Algebra of derivative

Theorem Let f and g be two functions which are differentiable at c. Then

(i) αf is differentiable at c where $\alpha \in \mathbb{R}$,

$$(\alpha f)'(c) = \alpha f'(c).$$

(ii) $f \pm g$ are differentiable at c and

$$(f \pm g)'(c) = f'(c) \pm g'(c)$$

(iii) fg is differentiable at c and

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c)$$

(iv) if $g(c) \neq 0$, $\frac{f}{g}$ is differentiable at c and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{\{g(c)\}^2}$$

~~(v) if $f(c) \neq 0$, $|f|$ is differentiable at c.~~

Proof

(i) Let $h = \alpha f$

Then for all $x \in \operatorname{dom} f$, $x \neq c$.

$$\frac{h(x) - h(c)}{x - c} = \frac{\alpha f(x) - \alpha f(c)}{x - c} = \frac{\alpha(f(x) - f(c))}{(x - c)}.$$

Since f is diff at $x = c$.

So $$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c).$$

$$\therefore \lim_{x \to c} \frac{h(x) - h(c)}{x - c} = \alpha . f'(c)$$

$$\therefore (\alpha f)'(c) = \alpha f'(c)$$

(ii) ~~For all $x \in D_f \cap D_g$~~

For all $x \in \operatorname{dom} f \cap \operatorname{dom} g$, $x \neq c$.

$$\frac{(f+g)(x) - (f+g)(c)}{x-c} = \frac{f(x)+g(x) - f(c) - g(c)}{x-c}$$

$$= \frac{f(x)-f(c)}{x-c} + \frac{g(x)-g(c)}{x-c}.$$

Since f is diff at c, $\lim\limits_{x\to c} \frac{f(x)-f(c)}{x-c} = f'(c)$.

Since g is diff at c, $\lim\limits_{x\to c} \frac{g(x)-g(c)}{x-c} = g'(c)$.

$$\therefore \lim_{x\to c} \frac{(f+g)(x)-(f+g)(c)}{x-c} = f'(c) + g'(c).$$

$$\therefore (f+g)'(c) = f'(c) + g'(c)$$

(iii) For all $x \in \operatorname{dom} f \cap \operatorname{dom} g$, $x \neq c$.

$$\frac{(fg)(x) - (fg)(c)}{x-c} = f(x) \cdot \frac{g(x)-g(c)}{x-c} + g(c)\,\frac{f(x)-f(c)}{x-c}.$$

As $x \to c$, $f(x) \to f(c)$ as f is continuous at c, by hypothesis.

As f is diff at c $\therefore$ $\lim\limits_{x\to c} \frac{f(x)-f(c)}{x-c} = f'(c)$.

As g is diff at c, $\therefore$ $\lim\limits_{x\to c} \frac{g(x)-g(c)}{x-c} = g'(c)$

$$\therefore \lim_{x\to c} \frac{(fg)(x)-(fg)(c)}{x-c} = f(c)\,g'(c) + g(c)\,f'(c).$$

$$\therefore (fg)'(c) = f(c)\,g'(c) + g(c)\,f'(c)$$

(iv) As $g(c) \neq 0$, $\exists$ an open interval I containing c such that $g(x) \neq 0$ in I, ~~becau~~ because g is continuous at c (by neighbourhood property of ~~a~~ continuity)

So for all $x \in \operatorname{dom}\left(\frac{f}{g}\right) \cap I$, $x \neq c$.

$$\frac{\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(c)}{x - c} = \frac{1}{g(x)g(c)}\left[g(c)\,\frac{f(x)-f(c)}{x-c} - f(c)\cdot\frac{g(x)-g(c)}{x-c}\right]$$

$g(x) \to g(c)$ as $x \to c$ ($\because$ g is continuous at c)

Also by hypothesis $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$

& $g'(c) = \lim_{x \to c} \frac{g(x)-g(c)}{x-c}$.

Hence $\lim_{x \to c} \frac{\frac{f}{g}(x) - \left(\frac{f}{g}\right)(c)}{x-c}$ exists and equal to

$$\frac{f'(c)\,g(c) - f(c)\,g'(c)}{\{\cancel{f}\,g(c)\}^2}$$

oblem

Show that $x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)} \quad \forall x > 0.$

Ans: We consider

$$f(x) = \log(1+x) - \left(x - \frac{x^2}{2}\right)$$

$$\therefore f'(x) = \frac{1}{1+x} - (1-x) = \frac{x^2}{1+x} > 0 \quad \forall x > 0$$

Hence f is increasing function for all $x > 0$.

Also $f(0) = \log(1+0) = \log 1 = 0$

Hence $f(x) > f(0) = 0$ for all $x > 0$

$$\Rightarrow \log(1+x) - \left(x - \frac{x^2}{2}\right) > 0$$

$$\Rightarrow \log(1+x) > x - \frac{x^2}{2} \quad \forall x > 0$$

Similarly we consider

$$F(x) = \log(1+x) - \left(x - \frac{x^2}{2(1+x)}\right)$$

$$= \frac{1}{1+x} - \left(1 - \frac{2x}{2(1+x)} - \frac{2x(1+x) - x^2}{2(1+x)^2}\right)$$

$$= \frac{1}{1+x} - 1 + \frac{x(1+x) + x(1+x) - x^2}{2(1+x)^2}$$

$$= \frac{1}{1+x} - 1 + \frac{2x + x^2}{(1+x)^2}$$

$$= \frac{(1+x - (1+x)^2 + 2x + x^2}{(1+x)^2}$$

$$= (1+$$

Similarly we consider

$$F(x) = \left(x - \frac{x^2}{2(1+x)}\right) - \log(1+x)$$

$$= 1 - \frac{2x}{1+x} - \frac{2x(1+x) - x^2}{2(1+x)^2} - \frac{1}{1+x}$$

$$= 1 + \frac{x}{1+x} - \frac{2x + x^2}{2(1+x)^2} - \frac{1}{1+x}$$

Darboux theorem

If f be derivable in a closed and bounded interval $[a,b]$ and $f'(a).f'(b) < 0$, then $\exists$ atleast one pt $c \in (a,b)$ such that

$$f'(c) = 0$$

Proof For the sake of definiteness, let us take

$$f'(a) > 0 \text{ and } f'(b) < 0.$$

f, being derivable in $[a,b]$, is continuous in $[a,b]$.

Therefore f will attain its maximum value at some pt c in $[a,b]$.

~~Let us define~~ $f'(a) > 0$. This implies f is increasing at a.

Therefore $\exists$ a $\delta > 0$ s.t $f(x) > f(a)$ $\forall x \in (a, a+\delta)$

This shows that $f(a)$ is not maximum value of f on $[a,b]$

Again $f'(b) < 0$. This implies f is decreasing at b.

Therefore $\exists$ a $\delta > 0$ such that $f(x) > f(b)$ $\forall x \in (b-\delta, b)$.

This shows that $f(b)$ is not maximum value of f on $[a,b]$.

Thus $c \neq a$, $c \neq b$ and therefore $a < c < b$

Since $c \in (a,b)$, $f'(c)$ exists. We prove that $f'(c) = 0$.

If $f'(c) > 0$, then $\exists$ $\delta > 0$ s.t $f(x) > f(c)$ for $x \in (c, c+\delta)$

This implies $f(c)$ is not maximium value of f on $[a,b]$

If $f'(c) < 0$, then $\exists$ $\delta > 0$ s.t $f(x) > f(c)$ for $x \in (c-\delta, c)$

This implies $f(c)$ is not maximum value of f on $[a,b]$

$$\therefore f'(c) = 0$$

This proves The Th

Intermediate value theorem for derivatives

If a function f is derivable on a closed & bounded interval $[a,b]$ and $f'(a) \neq f'(b)$ and K, a number lying between $f'(a)$ and $f'(b)$ then $\exists$ atleast one pt $c \in (a,b)$ such that $f'(c) = K$

Proof

Let $g(x) = f(x) - Kx \quad \forall x \in [a,b]$.

clearly g is derivable on $[a,b]$ and

$$g'(a) = f'(a) - K, \quad g'(b) = f'(b) - K$$

Since K lies between $f'(a)$ and $f'(b)$

$\therefore$ $g'(a)$ and $g'(b)$ are of opposite signs.

$\therefore$ $g(x)$ satisfies the conditions of Darboux Theorem.

Thus $\exists$ atleast one point $c \in (a,b)$ such that

$$g'(c) = 0$$

i.e $f'(c) = K$