Semester-II

Paper – C4T

Unit-1

- **Lipschitz (Cauchy-Lipschitz) Condition**

A function $f(x,y)$ is said to satisfy Lipschitz condition in a region R in xy-plane, if there exists a positive constant M, such that

$$|f(x,y_2)-f(x,y_1)| \leq M|y_2-y_1|$$

where the points (x,y_1) and (x,y_2) both lie in R. The constant M is known as Lipschitz constant for the corresponding function $f(x,y)$.

- **Picard's Theorem:** The initial value problem

$$\frac{dy}{dx}=f(x,y)\,;\quad y(0)=y_0 \qquad \text{———— (1)}$$

has a unique solution in the interval $|x-x_0|\leq h$, $h(>0)$ being any number, if $f(x,y)$ is continuous and has a continuous derivative $\frac{\partial f}{\partial y}$ in the closed rectangular domain D, where

$$\left[D : D[x_0-a\leq x\leq x_0+a,\; y_0-b\leq y\leq y_0+b]\right].$$

Geometrically, through the point (x_0,y_0), there passes a unique integral curve of the equation (1).

- **Linear Differential Equation of Order n**

A differential equation of the form

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_{n-1}\frac{dy}{dx} + P_n y = Q \quad \text{(2)}$$

Where $P_0, P_1, \cdots, P_n$ and Q are either constants or function of x (independent variable) is known as linear differential equation of order n where $P_0 \neq 0$.

- **Homogeneous differential equation**

If $Q = 0$, the equation (2) reduce to

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_n y = 0. \quad \text{(2)}$$

This equation is known as homogeneous differential equation.

- **Non-homogeneous differential equation**

If $Q \neq 0$, then the differential equation is called a non-homogeneous differential equation.

- **Homogeneous differential equation with constant-coefficient**

If $P_0, P_1, \ldots, P_n$ are all constants, then equation (2) is called homogeneous equation with constant coefficient.

Ex. 1. $3\frac{d^2 y}{dx^2} + 8\frac{dy}{dx} + 4y = 0$

Ex 2. $(D^2 + 4)y = 0$

Ex 3. $\frac{d^2 s}{dt^2} + s = 0$

- **General Solution of homogeneous equation of Second order.**

Let us consider the differential equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \quad \text{———— (1)}$$

where P and Q are constants.

Let $y = e^{mx} (\neq 0)$, be a trial solution of equation (1), then the auxiliary equation (A.E) will be

$$m^2 + Pm + Q = 0 \quad \text{———— (2)}$$

<u>Case I</u>: When the two roots of equation (2) are real and unequal say m_1 and m_2, then

$$G.S = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

where C_1 and C_2 are arbitrary constants.

<u>Case II</u>: When the roots of eqn (2) are real and equal say m_1 and m_1, then

$$G.S = (C_1 + C_2 x)\, e^{m_1 x}$$

where C_1 and C_2 are arbitrary constants.

<u>Case III</u>: When two roots of eqn (2) are complex, say $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, $i = \sqrt{-1}$ and α, β are two real numbers, then

$$G.S = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x),$$

C_1 and C_2 are two arbitrary constants.

Ex. Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$

Ans. The given equation is

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0 \quad \text{——— (1)}$$

Let $y = e^{mx} (\neq 0)$ be a trial solution of eq^n(1).

Then $\frac{dy}{dx} = me^{mx}$ and $\frac{d^2y}{dx^2} = m^2e^{mx}$

Putting these in eq^n(1) we get

$$e^{mx}(m^2+m-6) = 0$$

But $e^{mx} \neq 0$, so

$$m^2+m-6 = 0$$ [This eq^n is called auxiliary equation]

or, $(m-2)(m+3) = 0$

or, $m = 2, -3$

$\therefore$ The required general solution is $y = C_1e^{2x} + C_2e^{-3x}$, where C_1 and C_2 are arbitrary constants.

Ex. Solve $(D^2+8D+16)y = 0$

Ans. The given equation is

$$(D^2+8D+16)y = 0 \quad \text{——— (1)}$$

Let $y = e^{mx} (\neq 0)$ be a trial solution of eq^n(1).

Then the A.E. is

$$m^2+8m+16 = 0$$

or, $(m+4)^2 = 0$

or, $m = -4, -4$

$\therefore$ The required general solution is $y = (C_1+C_2x)e^{-4x}$, where C_1 and C_2 are arbitrary constants.

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Ex. Solve $y'' + 8y' + 25y = 0$, where $y' = \frac{dy}{dt}$

Ans. The given equation is

$$y'' + 8y' + 25y = 0 \qquad \text{———— (1)}$$

Let $y = e^{mt}\ (\neq 0)$ be a trial solution of eqn (1)

Therefore the A.E. is

$$m^2 + 8m + 25 = 0$$

$$\text{or,}\quad m = \frac{-8 \pm \sqrt{64 - 100}}{2}$$

$$= \frac{-8 \pm \sqrt{-36}}{2}$$

$$= \frac{-8 \pm 6i}{2}$$

$$= -4 \pm 3i$$

∴ The required general solution is

$y = e^{-4t}(c_1 \cos 3t + c_2 \sin 3t)$, where c_1 and c_2 are arbitrary constants.

Assignment

Solve:

1. $\frac{d^2y}{dx^2} + 9y = 0$

2. $\frac{d^2p}{ds^2} - 2\frac{dp}{ds} = 0$

3. $y'' + 2y' + y = 0$, $y' = \frac{dy}{dx}$

4. $(D^2 + 7D - 8)y = 0$

5. $r'' - r = 0$, $r' = \frac{dr}{dt}$