Concavity, convexity, point of inflexion

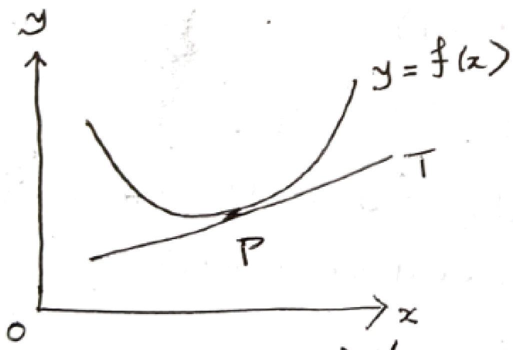
Consider a plane curve whose equation w.r.to a given set of rectangular axes be $y = f(x)$. Let P be a point on this curve. We assume that the tangent PT at this pt P is not parallel to y axis. Then if the curve does not cross its tangent at P, it will, before and after the pt, be situated on the same side of the tangent PT in a small neighbourhood of P.

That is, there exists a sufficiently small arc of the curve which i) lies entirely above the tangent
ii) lies entirely below the tangent

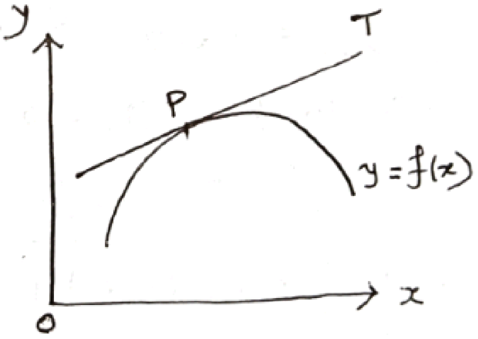
In case (i), the curve is said to be concave upwords at P (or equivalently, convex downwards at P)

In case (ii), the curve is said to be concave downwards at P (or equivalently, convex upwards at P)

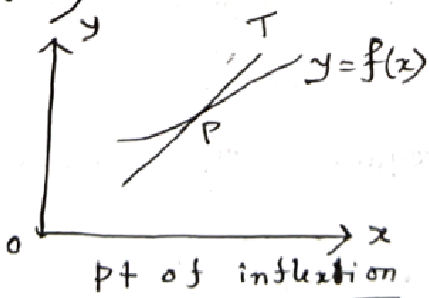
We note that, if the curve crosses the tangent at P, we say the curve has a point of inflexion at P.



(concave upwards / convex ~~upw~~ downwards)



(concave downwards or convex upwards)



pt of inflexion

Criterion for Concavity or Convexity

Suppose the derivatives of first two orders $f'(x)$ & $f''(x)$ exist and are continuous in a small neighbourhood of the pt $P(x,y)$ where $f''(x) \neq 0$.

Then the curve will be i) Concave upwards (or convex downwards) if $f''(x) > 0$

and ii) concave downwards (or convex upwards) if $f''(x) < 0$

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The point of inflexion of the curve $y = f(x)$ are those roots of $f''(x) = 0$ where $f''(x)$ has one sign in the left neighbourhood and an opposite sign in the right neighbourhood of the points of consideration. (i.e $f'''(x) \neq 0$)

Problem 1.

Test for concavity and existance of points(s) of inflexion of the curve $y = e^{-x^2}$

Ans:

$$y_1 = -2x e^{-x^2}$$

$$y_2 = 2e^{-x^2}(2x^2-1)$$

We note that $y_2 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$.

when $-\infty < x < -\frac{1}{\sqrt{2}}$, $y_2 > 0$ and the curve is concave upwards (or equivalently convex downwards)

when $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$, $y_2 < 0$ and the curve is concave downwards (or equivalently convex upwards)

when $\frac{1}{\sqrt{2}} < x < \infty$, $y_2 > 0$ and the curve is concave upwards (or equivalently convex downwards)

We note that the 2nd order derivative changes sign when passing through $x = -\frac{1}{\sqrt{2}}$. The corresponding point of inflexion is $\left(-\frac{1}{\sqrt{2}}, e^{-1/2}\right)$.

Due to similar reason, the curve has another point of inflexion is $\left(\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$.

Problem 2. At what values of 'a' and 'b' does the point (1, 3) serve as the point of inflexion of the curve $y = ax^3 + bx^2$?

Ans:
$$y_1 = 3ax^2 + 2bx$$
$$y_2 = 6ax + 2b$$

At (1, 3), $y_2 = 6a + 2b$.

As y_2 exists, so at point of inflexion (1, 3), $y_2 = 0$

$\Rightarrow 6a + 2b = 0$ or, $b = -3a$ —— ①

The point (1, 3) is on the curve & so $a + b = 3$ —— ②

We get (from ① & ②)

$$a = -\frac{3}{2}, \quad b = \frac{9}{2}$$

$y_3 = 6a \neq 0$. So when $a = -\frac{3}{2}$, $b = \frac{9}{2}$, (1, 3) is a point of ~~infex~~ inflexion.

Problem 3. Show that the point $\left(-2, -\frac{2}{e^2}\right)$ is a point of inflexion of the curve $y = xe^x$.

Ans: $y_1 = xe^x + e^x$, $y_2 = xe^x + 2e^x$, $y_3 = xe^x + 3e^x$

$y_2 = 0 \Rightarrow x = -2$ and so from the equ^n of the curve $y = -\frac{2}{e^2}$

For $x = -2$, $y_3 = e^{-2}(3-2) \neq 0$.

So the pt $\left(-2, -\frac{2}{e^2}\right)$ is a point of inflexion.

~~Problem 4. Prove that the curve $y = \log x$ $(x > 0)$ is everywh convex upwards. Discuss concavity or convexity with respect~~

Problem 4. Find the intervals where the curve $y = e^x(\cos x + \sin x)$ is concave upwards or downwards for $0 < x < 2\pi$.

Ans

Here $y_1 = \dfrac{dy}{dx} = e^x(\cos x + \sin x) + e^x(\cos x - \sin x)$

$$= 2e^x \cos x.$$

$$y_2 = \frac{d^2y}{dx^2} = 2e^x(\cos x - \sin x)$$

Now $\dfrac{d^2y}{dx^2} > 0$ implies $\cos x - \sin x > 0$

i.e $\cos x > \sin x$

Now $\cos x > \sin x$ holds when $0 < x < \pi/4$, $\frac{5\pi}{4} < x < 2\pi$

Thus the curve is concave upwards in $(0, \pi/4) \cup (\frac{5\pi}{4}, 2\pi)$

Again, $\dfrac{d^2y}{dx^2} < 0$ implies $\cos x - \sin x < 0$

i.e $\cos x < \sin x$.

Now $\cos x < \sin x$ holds when $\frac{\pi}{4} < x < \frac{5\pi}{4}$

Thus the curve is concave downwards for $(\pi/4, \frac{5\pi}{4})$

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Problem 5:

Show that the points of inflexion of the curve $y^2 = (x-a)^2(x-b)$ lie on the line $3x + a = 4b$

Ans:

The equ^n of the curve is $y^2 = (x-a)^2(x-b)$ — (1)

or, $y = (x-a)(x-b)^{1/2}$

or, $$\frac{dy}{dx} = \frac{2(x-b) + (x-a)}{\sqrt{x-b}}$$

$$= \frac{3x - 2b - a}{\sqrt{x-b}}$$

$$\frac{d^2y}{dx^2} = \frac{3\sqrt{x-b} - (3x-2b-a)\cdot\frac{1}{2}\frac{1}{\sqrt{x-b}}}{(x-b)}$$

$$= \frac{6(x-b) - (3x-2b-a)}{2(x-b)^{3/2}}$$

$$= \frac{3x - 4b + a}{2(x-b)^{3/2}}$$

For the pt of inflexion, $\frac{d^2y}{dx^2} = 0$

$$\Rightarrow x = \frac{4b-a}{3}$$

$$\therefore y = \left(\frac{4b-a}{3} - a\right)\left(\frac{4b-a}{3} - b\right)^{1/2}$$

$$= 4\frac{(b-a)}{3}\frac{(b-a)^{1/2}}{\sqrt{3}}$$

$$= 4\left(\frac{b-a}{3}\right)^{3/2}$$

Now $$\frac{d^2y}{dx^2} = \frac{3x-4b+a}{2(x-b)^{3/2}} > 0 \text{ if } x > \frac{4b-a}{3}$$

$$< 0 \text{ if } x < \frac{4b-a}{3}$$

$\therefore \left(\frac{4b-a}{3}, 4\left(\frac{b-a}{3}\right)^{3/2}\right)$ is a pt of inflexion.

Clearly the pt lie on the line $3x + a = 4b$

Problem 6.

Prove that the line joining the points of inflexion of the curve $y^2(x-a) = x^2(x+a)$ subtend an angle of $\frac{\pi}{3}$ at the origin.

Sol^n Here, $y = \dfrac{x\sqrt{x+a}}{\sqrt{x-a}}$

Now, $\dfrac{dy}{dx} = \dfrac{x^2-ax-a^2}{(x-a)\sqrt{x^2-a^2}}$

and $\dfrac{d^2y}{dx^2} = \dfrac{(x^2-a^2)(2x-a)-(x^2-ax-a^2)(2x+a)}{(x-a)(x^2-a^2)}$

Now, the pts of inflexion are at those pts where $\dfrac{d^2y}{dx^2} = 0$

Hence

$$(x^2-a^2)(2x-a)-(x^2-ax-a^2)(2x+a) = 0$$

or, $x+2a = 0$ i.e $x = -2a$

At $x = -2a$, $y = \pm\dfrac{2a}{\sqrt{3}}$.

Note: For points of the curve $y = f(x)$, the points where the tangent is parallel to the y axis (i.e $\frac{dy}{dx}$ is undefined) it becomes necessary to investigate $\frac{d^2x}{dy^2}$ instead of $\frac{d^2y}{dx^2}$

Note 1. A curve $y = f(x)$ is convex or concave at $P(x,y)$ w.r. to the axis of x (on foot of the ordinate) according as

$$y \frac{d^2y}{dx^2} \text{ is positive or negative}$$

2. A curve $x = f(y)$ is convex or concave at $P(x,y)$ w.r. to axis of y (on foot of the abscissa) according as

$$x \frac{d^2x}{dy^2} \text{ is positive or negative.}$$

Problem 7.

Prove that the curve $(y-a)^3 = a^3 - 2a^2x + ax^2$, $(a>0)$ is always concave to the x-axis.

Sol^n Here

$$(y-a)^3 = a^3 - 2a^2x + ax^2$$
$$= a(x-a)^2$$

Taking logarithm to both sides, we get

$$3\log(y-a) = \log a + 2\log(x-a)$$

Differentiating both sides with respect to x it follows that

$$\frac{dy}{dx} = \frac{2}{3}\,\frac{y-a}{x-a}$$

Also

$$\frac{d^2y}{dx^2} = -\frac{2}{9}\,\frac{a(y-a)}{(x-a)^2} = -\frac{2}{9}\,\frac{(y-a)\cdot a}{(y-a)^3}$$
$$= -\frac{2}{9}\,\frac{a}{(y-a)^2}$$

Now, $y = a + a^{1/3}(x-a)^{2/3}$

$\therefore \quad y\frac{d^2y}{dx^2} = -\frac{2}{9}\frac{a}{(y-a)^2}\left[a + a^{1/3}(x-a)^{2/3}\right]$

Hence $y\frac{d^2y}{dx^2} < 0$ for all points as $(x-a)^{2/3}$ is positive always.

Thus the curve is concave to the x-axis always.

Problem 8.

Find the range of values of x for which the curve $y = x^4 - 16x^3 + 42x^2 + 12x + 1$ is concave or convex ~~with respect to the x-axis~~ and identify the pts of inflexion.

Sol^n:

Here

$$y = x^4 - 16x^3 + 42x^2 + 12x + 1$$

$$\therefore \frac{dy}{dx} = 4x^3 - 48x^2 + 84x + 12$$

$$\frac{d^2y}{dx^2} = 12x^2 - 96x + 84$$

$$= 12(x-1)(x-7)$$

and $$\frac{d^3y}{dx^3} = 24x - 96$$

Now for

$x < 1$, $\frac{d^2y}{dx^2} > 0$

$x = 1$, $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} \neq 0$

$1 < x < 7$, $\frac{d^2y}{dx^2} < 0$

$x = 7$, $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} \neq 0$

and $x > 7$, $\frac{d^2y}{dx^2} > 0$

Thus The curve is concave upwards in the interval $(-\infty, 1) \cup (7, \infty)$ and concave downwards in $(1, 7)$

Thus at $x = 1, x = 7$, the curve has point of inflexion

Now at $x = 1, y = 90$ and $x = 7, y = -944$

Hence $(1, 90), (7, -944)$ are the points of inflexion.

Problem 9

Prove that the curve $y = \log x \ (x > 0)$ is everywhere upwards. Discuss cancavity or convexity with respect to x-axis.

Sol^n

Here $y = \log x \quad (x > 0)$

$$\frac{dy}{dx} = \frac{1}{x}, \quad \frac{d^2y}{dx^2} = -\frac{1}{x^2}$$

Hence $\frac{d^2y}{dx^2} < 0$ for all values of $x > 0$

Thus The curve is concave downwards at all pts where $x > 0$.

Now $y = \log x$ is negative for $0 < x < 1$ and positive for $x > 1$

Hence for $0 < x < 1$, $y\frac{d^2y}{dx^2} > 0$ and for $x > 1$,

$$y\frac{d^2y}{dx^2} < 0$$

Thus the curve is Convex w.r.to x-axis if $0 < x < 1$

and Concave w.r.to x-axis if $x > 1$

Problem 10

Prove that the curve $y = e^{2x}$ is convex to The x-axis at every pt

Sol^n

$$\frac{dy}{dx} = 2e^{2x}$$

$$\frac{d^2y}{dx^2} = 4e^{2x}$$

$$y\frac{d^2y}{dx^2} = 4e^{2x} \cdot e^{2x} = 4e^{4x} > 0 \quad \forall x$$

Thus the curve is convex to the x-axis at every pt.

Problem 11

Show that the origin is a point of inflexion on the curve $y = x^2 \log(1-x)$.

Solⁿ:

$$\frac{dy}{dx} = 2x\log(1-x) - \frac{x^2}{1-x}.$$

$$\frac{d^2y}{dx^2} = 2\log(1-x) - \frac{4x}{1-x} - \frac{x^2}{(1-x)^2}$$

$$\frac{d^3y}{dx^3} = -\frac{2}{1-x} - \frac{4(1-x)+4x}{(1-x)^2} - \frac{(1-x)^2 \cdot 2x + x^2 2(1-x)}{(1-x)^4}$$

$$= \frac{-2}{1-x} - \frac{4}{(1-x)^2} - \frac{2x - 4x^2 + 2x^3 + 2x^2 - 2x^3}{(1-x)^4}$$

$$= \frac{-2}{1-x} - \frac{4}{(1-x)^2} - \frac{2x-2x^2}{(1-x)^4}$$

$$= -\frac{2}{1-x} - \frac{4}{(1-x)^2} - \frac{2x}{(1-x)^3}$$

At $(0,0)$ $\quad \frac{d^2y}{dx^2} = 0$ But $\frac{d^3y}{dx^3} = -6 \neq 0$

Hence $(0,0)$ is a point of inflexion

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Problem 12

Prove that the curve $y = 2\sqrt{ax}$ is concave to the foot of the ordinate everywhere except the origin.

sol^n

$$\frac{dy}{dx} = \sqrt{\frac{a}{x}}.$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2}\frac{\sqrt{a}}{x^{3/2}}$$

$$y\frac{d^2y}{dx^2} = -\frac{a}{x}.$$

Hence, for $x>0$, $y\frac{d^2y}{dx^2} < 0$

Here y is not defined for $x<0$.

Again at $x=0$, $y\frac{d^2y}{dx^2}$ is not defined.

Hence the curve is concave everywhere to the foot of the ordinate except the origin.

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Homework

1. Show that the pts of inflexion of the curve $y = \frac{\sin x}{x}$ lie on the curve $y^2(4+x^4) = 4$

2. Test for concavity / convexity & pts of inflexion of the curve $y = b^3/(a^2+x^2)$

3. Prove that the curve $y = \log x\ (x>0)$ is convex everywhere to the y-axis.

4. Show that the curve $y^3 = 27x^2$ is concave to the foot of the ordinate everywhere at the origin.

5. If the function $f(x) = ax^3 + bx^2$ has a pt of inflexion at $(1,2)$ find the values a & b.

6. Prove that $(a-2, -2/e^2)$ is a pt of inflexion of the curve $y = (x-a)e^{x-a}$.

7. Show that the curve $(a^2+x^2)y = a^2x$ has three points of inflexion.

8. Show that every point in which the curve $y = c\sin(x/a)$ meets the x-axis is a pt of inflexion.
