Example 5.9 Find the magnetic field of a very long solenoid, consisting of n closely wound turns per unit length on a cylinder of radius R and carrying a steady current I. (The point of making the windings so close is that one can then pretend each turn is circular.)

ans:- First of all, we must understand that $\vec{B}$ doesnot have a radial component. This is so because -- for suppose B_r were positive, if we reversed the direction of the current, B_r would then be negative. But switching I is physically equivalent to turning the solenoid upside down, and that certainly should not alter the radial field. There should also be no circumferential component of $\vec{B}$. For B_φ would be constant around an amperian loop concentric with the solenoid and hence,

$$\oint \vec{B}\cdot d\vec{l} = B_\varphi \cdot (2\pi r) = \mu_0 I_{enc} = 0.$$

since the loop encloses no current.

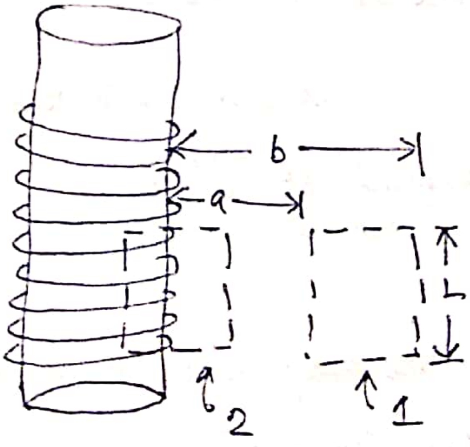
So the magnetic field of an infinite, closely wound solenoid runs parallel to the axis. From the right hand rule, we expect that it points upwards inside the solenoid and downward outside. Moreover, it certainly approaches zero as one goes very far away.

Applying Ampere's law to the two rectangular loop 1 & 2 as shown in the figure alongside

— (For loop I)

$$\oint \vec{B}\cdot d\vec{l} = [B(a) - B(b)]L = \mu_0 I_{enc} = 0.$$

(Loop I lies entirely outside the solenoid, with its sides at distances a and b from the axis), or, $B(a) = B(b)$.



Amperian loop.

Evidently the field outside doesnot depend on the distance from the axis. But we know that it goes to zero for large r. It must therefore be zero everywhere.

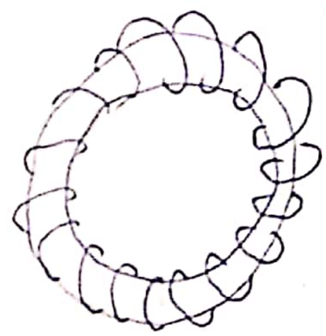
For loop 2, which is half inside and half outside, Ampere's law gives –

$$\oint \vec{B} \cdot d\vec{l} = BL = \mu_0 I_{enc} = \mu_0 n I L .$$

where B is the field inside the solenoid. (The right side of the loop contributes nothing, since $B=0$ outside).

$$\therefore \quad \vec{B} = \begin{cases} \mu_0 n I \hat{z} & \text{inside the solenoid} \\ 0 & \text{outside the solenoid} \end{cases}$$

Example 5.10 A toroidal coil consists of a circular ring, or 'donut' around which a long wire is wrapped. The winding is uniform and tight enough so that each turn can be considered a closed loop. The cross sectional shape of the coil is immaterial. Just as long as the shape remains the same all the way around the ring, the magnetic field of the toroid is circumferential at all points, both inside and outside the coil.



According to Biot-Savart's law, the field at $\vec{r}$ due to the current element at $\vec{r}'$ is –

$$d\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{\vec{I} \times \vec{\mathscr{r}}}{\mathscr{r}^3} dl' .$$

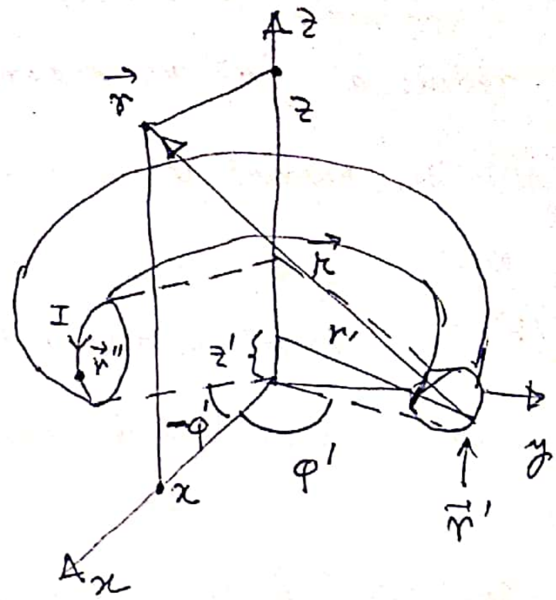
One may put $\vec{r}$ in the xz plane, so that, its cartesian components are $(x, 0, z)$, while the source coordinates are $\vec{r}' = (r'\cos\varphi', r'\sin\varphi', z)$

Then, $\vec{r} = (x - r'\cos\varphi', -r'\sin\varphi', z - z')$

Since the current has no ϕ component,

$\vec{I} = I_r \hat{r} + I_z \hat{z}$, or in cartesian coords.,

$\vec{I} = (I_r \cos\varphi', I_r \sin\varphi', I_z)$



Accordingly,

$$\vec{I} \times \vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ I_r\cos\varphi' & I_r\sin\varphi' & I_z \\ (x - r'\cos\varphi') & (-r'\sin\varphi') & (z - z') \end{vmatrix}$$

$$= [\sin\varphi'(I_r(z-z') + r' I_z)]\hat{x} + [I_z(x - r'\cos\varphi') - I_r\cos\varphi'(z-z')]\hat{y}$$

$$+ [-I_r x \sin\varphi']\hat{z}.$$

But there is a symmetrically situated current element at $\vec{r}''$, with the same r', the same r, the same dl', the same I_r, and the same I_z, but negative φ'. Because $\sin\varphi'$ changes sign, the $\hat{x}$ and $\hat{z}$ contributions from $\vec{r}'$ and $\vec{r}''$ cancels, leaving a $\hat{y}$ term. Thus the field at $\vec{r}$ is in the $\hat{y}$ direction, and in general the field points in the $\hat{\varphi}$ direction. Since the field is circumferential, applying Ampere's law to a circle of radius s about the axis of the toroid:-

$$B \cdot 2\pi s = \mu_0 I_{enc}$$

Hence,

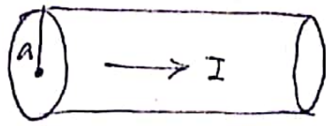
$$\vec{B}(\vec{r}) = \begin{cases} \dfrac{\mu_0 N I}{2\pi s}\hat{\varphi} & \text{for points inside the coil} \\ 0 & \text{for points outside the coil.} \end{cases}$$

and N is the total number of turns.

Problem 5.13 A steady current I flows down a long cylindrical wire of radius a. Find the magnetic field both inside and outside the wire, if-

(a) The current is uniformly distributed over the outside surface of the wire.

(b) The current is distributed in such a way that J is proportional to r, the distance from the axis.



ans:-(a) $\oint \vec{B}\cdot d\vec{l} = B \cdot 2\pi r = \mu_0 I_{enc}$

so, $B = \begin{cases} 0 & \text{for } r < a \\ \dfrac{\mu_0 I}{2\pi r}\hat{\varphi} & \text{for } r > a \end{cases}$

(b) $J = kr$, $I = \int_0^a J\,da = \int_0^a kr \cdot 2\pi r\,dr = \dfrac{2\pi k a^3}{3}$

so, $k = \dfrac{3I}{2\pi a^3}$

$\therefore I_{enc} = \int_0^a J\,da = \int_0^r kr'(2\pi r')\,dr'$

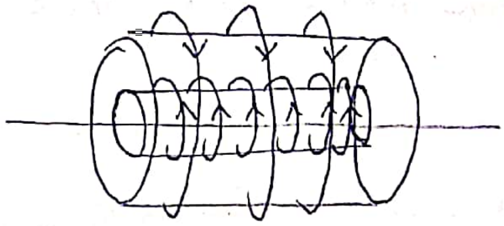
$= \dfrac{2\pi k r^3}{3}$ $\quad$ ($\because$ putting $k = \dfrac{3I}{2\pi a^3}$).

$= \dfrac{I r^3}{a^3}$.

for $r<a$; $I_{enc} = I$, for $r>a$, So,

$$B = \begin{cases} \dfrac{\mu_0 I r^2}{2\pi a^3}\hat{\varphi} & \text{for } r<a \\ \dfrac{\mu_0 I}{2\pi r}\hat{\varphi} & \text{for } r>a. \end{cases}$$

Problem 5.15 Two long coaxial solenoids each carry current I, but in ~~oppos~~ opposite directions, as shown in the figure. The inner solenoid (radius a) has n_1 turns per unit length, and the outer one (radius b) has n_2. Find $\vec{B}$ in each of the three regions: (i) inside the inner solenoid (ii) between them, (iii) outside both.



ans:- The field inside a solenoid is $\mu_0 n I$, and outside it is zero. The outer solenoid's field points to the right ($\hat{z}$), whereas the inner ones points to the left ($-\hat{z}$). So,

(i) $\vec{B} = \mu_0 I (n_2 - n_1)\hat{z}$

(ii) $\vec{B} = \mu_0 I n_2 \hat{z}$

(iii) $\vec{B} = 0$.

Problem 5.20 Is Ampere's circuital law consistent with the general rule that divergence of curl is always zero? Show that Ampere's law cannot be valid, in general, outside magnetostatics.

ans:- From Ampere's circuital law, $\nabla \times \vec{B} = \mu_0 \vec{J}$.

$$\therefore \nabla \cdot (\nabla \times \vec{B}) = \mu_0 (\nabla \cdot \vec{J}).$$

But, from continuity equation, $\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$, so,

$$\nabla \cdot (\nabla \times \vec{B}) = -\mu_0 \frac{\partial \rho}{\partial t}.$$

This is inconsistent with $\nabla \cdot (\nabla \times \vec{B}) = 0$ unless ρ = constant. Hence, this is valid only for magnetostatics.

Magnetic vector potential

$\nabla \cdot \vec{B} = 0$, so, we can introduce a vector potential $\vec{A}$ in magnetostatics, $\vec{B} = \nabla \times \vec{A}$. This formulation of vector potential automatically takes care of $\nabla \cdot \vec{B} = 0$, since the divergence of curl is always zero.

From Ampere's law, $\nabla \times \vec{B} = \nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$. — ①.

Now, we can add to the magnetic potential any function whose curl vanishes (which is to say, the gradient of any scalar), with no effect on $\vec{B}$. We can use this to eleminate the divergence of $\vec{A}$:

$$\nabla \cdot \vec{A} = 0 \quad — ②.$$

Now, using ② in ①, Ampere's law becomes — $\boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$ — ③.

Now, to prove that $\nabla \cdot \vec{A} = 0$ is always possible, suppose that our original potential $\vec{A}_0$ is not divergenceless. If we add to it the gradient of λ ($\vec{A} = \vec{A}_0 + \nabla \lambda$), the new divergence is — $\nabla \cdot \vec{A} = \nabla \cdot \vec{A}_0 + \nabla^2 \lambda$. This would satisfy eqn. ②, if we can set $\nabla^2 \lambda = -\nabla \cdot \vec{A}_0$. This is mathematically identical to Poisson's equation — $\nabla^2 V = -\rho/\epsilon_0$, with $\nabla \cdot \vec{A}_0$ in place of ρ/ϵ_0 as the 'source'. The solution to Poisson's eqn. is,

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho}{r} d\tau' .$$

and in the same manner, if $\nabla \cdot \vec{A}_0$ goes to zero at infinity, then —

$$\lambda = \frac{1}{4\pi} \int \frac{\nabla \cdot \vec{A}_0}{r} d\tau'$$

If $\nabla \cdot \vec{A}_0$ doesnot go to zero at infinity, we can always find means to find the appropriate λ. Hence, it is always possible to make the vector potential divergenceless. In other words, the definition $\vec{B} = \nabla\times\vec{A}$ specifies the curl of $\vec{A}$, but it doesnot say anything about the divergence — we are at liberty to pick as we see fit, and zero is ordinarily the simplest choice. with this condition on $\vec{A}$, ampere's law becomes —

$$\nabla^2\vec{A} = -\mu_0\vec{J}.$$

Assuming $\vec{J}$ goes to zero at infinity, $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}')}{r}\,d\tau'$

<u>Example 5.12</u> Find the vector potential of a infinite solenoid with n turns per unit length, radius R, and current I.

<u>ans</u>:-

$$\oint\vec{A}\cdot\vec{dl} = \int(\nabla\times\vec{A})\cdot\vec{da} = \int\vec{B}\cdot\vec{da} = \varphi.$$

where φ is the flux of $\vec{B}$ through the loop in question. This is reminiscent of Ampere's law in integral form — $\oint\vec{B}\cdot\vec{dl} = \mu_0 I_{enc}$. Infact, it's the same equation, with $\vec{B}\to\vec{A}$ and $\mu_0 I_{enc}\to\varphi$. The present problem of a uniform longitudinal magnetic field $\mu_0 nI$ inside the solenoid and no field outside is analogous to the Ampere's law problem of a fat wire carrying a uniformly distributed current. The vector potential is circumferential like the magnetic field of the wire; using a circular amperian loop of radius s inside the solenoid, we have —

$$\oint\vec{A}\cdot\vec{dl} = A\cdot 2\pi s = \int\vec{B}\cdot\vec{da} = \mu_0 nI(\pi s^2).$$

or, $\vec{A} = \frac{\mu_0 nIs}{2}\hat{\varphi}$, for $s<R$.

For an amperian loop outside the solenoid, the flux is –

$$\int \vec{B} \cdot d\vec{a} = \mu_0 n I \cdot \pi R^2$$

since the field only extends out to R. Thus,

$$\vec{A} = \frac{\mu_0 n I}{2} \cdot \frac{R^2}{s} \hat{\varphi}, \quad \text{for } s > R.$$

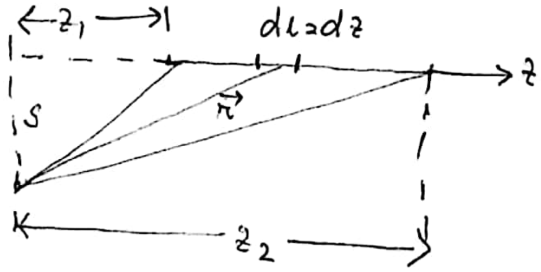
<u>Problem 5.22</u> Find the magnetic vector potential of a finite segment of straight wire, carrying a current I. Put the wire on z-axis, from z_1 to z_2.

<u>ans:</u>–

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{I}{\imath} \hat{z}\, dz = \frac{\mu_0 I}{4\pi} \hat{z} \int_{z_1}^{z_2} \frac{dz}{\sqrt{z^2+s^2}}$$

$$= \frac{\mu_0 I}{4\pi} \hat{z} \left[\ln\left(z + \sqrt{z^2+s^2}\right) \right]_{z_1}^{z_2}$$

$$= \frac{\mu_0 I}{4\pi} \ln\left[\frac{z_2 + \sqrt{z_2^2+s^2}}{z_1 + \sqrt{z_1^2+s^2}} \right] \hat{z}$$



$$\vec{B} = \nabla \times \vec{A} = -\frac{\partial A}{\partial s} \hat{\varphi}$$

$$= -\frac{\mu_0 I}{4\pi} \left[\frac{1}{z_2 + \sqrt{z_2^2+s^2}} \cdot \frac{s}{\sqrt{z_2^2+s^2}} - \frac{1}{z_1 + \sqrt{z_1^2+s^2}} \cdot \frac{s}{\sqrt{z_1^2+s^2}} \right] \hat{\varphi}.$$

$$= -\frac{\mu_0 I s}{4\pi} \left[\frac{z_2 - \sqrt{z_2^2+s^2}}{z_2^2 - (z_2^2+s^2)} \cdot \frac{1}{\sqrt{z_2^2+s^2}} - \frac{z_1 - \sqrt{z_1^2+s^2}}{z_1^2 - (z_1^2+s^2)} \cdot \frac{1}{\sqrt{z_1^2+s^2}} \right] \hat{\varphi}.$$

$$= -\frac{\mu_0 I s}{4\pi} \left(-\frac{1}{s^2}\right) \left[\frac{z_2}{\sqrt{z_2^2+s^2}} - 1 - \frac{z_1}{\sqrt{z_1^2+s^2}} + 1 \right] \hat{\varphi}$$

$$= \frac{\mu_0 I}{4\pi s} \left[\frac{z_2}{\sqrt{z_2^2+s^2}} - \frac{z_1}{\sqrt{z_1^2+s^2}} \right] \hat{\varphi}$$

or, since, $\sin\theta_1 = \frac{z_1}{\sqrt{z_1^2+s^2}}$ and $\sin\theta_2 = \frac{z_2}{\sqrt{z_2^2+s^2}}$

$$\therefore\ \vec{B} = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1)\,\hat{\varphi}.$$

Problem 5.23 What current density would produce the vector potential, $\vec{A} = k\hat{\phi}$ (where k is a constant), in cylindrical coordinates?

ans:- $A_\varphi = k$

$$\vec{B} = \nabla\times\vec{A} = \frac{1}{s}\frac{\partial}{\partial s}(sk)\hat{z} = \frac{k}{s}\hat{z}.$$

$$\vec{J} = \frac{1}{\mu_0}(\nabla\times\vec{B}) = \frac{1}{\mu_0}\left[-\frac{\partial}{\partial s}\left(\frac{k}{s}\right)\right]\hat{\varphi} = \frac{k}{\mu_0 s^2}\hat{\varphi}.$$

Problem 5.24 If $\vec{B}$ is uniform, show that $\vec{A}(\vec{r}) = -\frac{1}{2}(\vec{r}\times\vec{B})$ works. That is check that $\nabla\cdot\vec{A} = 0$ and $\nabla\times\vec{A} = \vec{B}$. Is this result unique? or are there other functions with the same divergence and curl?

ans:- $\nabla\cdot\vec{A} = -\frac{1}{2}\nabla\cdot(\vec{r}\times\vec{B}) = -\frac{1}{2}[\vec{B}\cdot(\nabla\times\vec{r}) - \vec{r}\cdot(\nabla\times\vec{B})] = 0,$

since $\nabla\times\vec{B} = 0$ ($\vec{B}$ is uniform) and $\nabla\times\vec{r} = 0$.

$$\nabla\times\vec{A} = -\frac{1}{2}\nabla\times(\vec{r}\times\vec{B}) = -\frac{1}{2}\Big[(\vec{B}\cdot\nabla)\vec{r} - (r\cdot\nabla)\vec{B} + \vec{r}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{r})\Big].$$

But $(\vec{r}\cdot\nabla)\vec{B} = 0$ and $\nabla\cdot\vec{B} = 0$ (since $\vec{B}$ is uniform), and $\nabla\cdot\vec{r} = 3$.

$$(\vec{B}\cdot\nabla)\vec{r} = \left(B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}\right)(x\hat{x} + y\hat{y} + z\hat{z}) = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$$
$$= \vec{B}.$$

So, $(\nabla\times\vec{A}) = -\frac{1}{2}(\vec{B} - 3\vec{B}) = \vec{B}$.

Magnetic force on a charge -

$\boxed{\vec{F} = q(\vec{v} \times \vec{B})}$ for a charge q.

$|\vec{F}| = q|\vec{v}||\vec{B}|\sin\theta$. If $\vec{v} \perp \vec{B}$, then $\theta = 90^\circ$ and $\sin\theta = 1$.

Then, $\boxed{F = qvB}$.

Magnetic force on a current carrying wire -

The magnetic force on a moving charge is, $F = qvB\sin\theta$.

If we use the definition of current, as the amount of charge q passing a point in a certain time interval Δt and expressing velocity v as a length L divided by a time interval Δt,

$$F = qvB\sin\theta = q.\frac{L}{\Delta t}.B\sin\theta = ILB\sin\theta.$$

$\therefore$ The magnitude of the magnetic force exerted on a wire of length L, carrying current I, by a magnetic field of magnitude B is

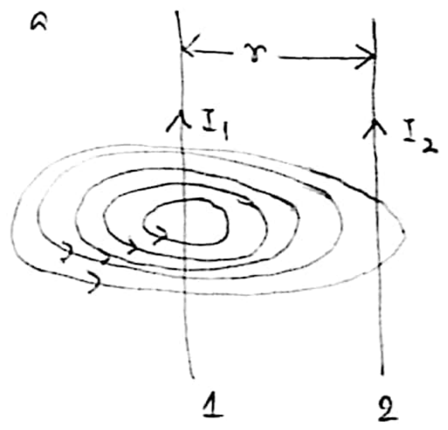
$$\boxed{F = ILB\sin\theta}.$$

where θ is the angle between the current direction and magnetic field. The direction of the force is given by a right hand rule that follows the rule for charges. Point the fingers on your right hand in the direction of the current. Align your hand so that when you curl your fingers, they point in the direction of the magnetic field. Stick out your thumb, and it points in the direction of the magnetic force. The current direction is defined to be the direction of flow of positive charge, so the direction given by your hand never needs to be reversed, as it does when we find the direction of the magnetic force on a negatively charged particle.

Magnetic force between two parallel current carrying wire –

The force between two long, straight and parallel conductors separated by a distance r can be found by applying the field due to a long wire. Let us consider the field produced by wire 1 and the force it exerts on wire 2. The field due to I_1 at a distance r is –

$$B_1 = \frac{\mu_0 I_1}{2\pi r} \quad \text{—(1)}$$



This field is uniform from the wire 1 and ⊥r to it, so the force F_2 it exerts on a length l of wire 2 is given by $F = ILB \sin\theta$. with $\theta = 90°$, $\sin\theta = 1$.

$$\therefore \quad F_2 = I_2 l B_1 \quad \text{—(2)}$$

The forces on the wires are equal ~~as~~ in magnitude, so we just write F for the magnitude of $\vec{F_2}$ ($\vec{F_1} = -\vec{F_2}$). Since the wires are very long, it is convinient to think in terms of F/l, the force per unit length. Substituting the ~~force~~ expression for B_1 into (2) and rearranging terms, we get,

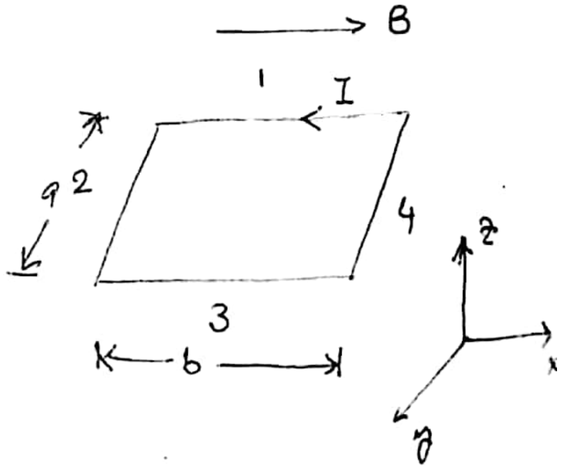
$$\frac{F}{l} = \frac{\mu_0 I_1 I_2}{2\pi r}$$

The ratio F/l is the force per unit length between two parallel currents I_1 and I_2 separated by a distance r. The force is attractive if the currents are in the same direction and repulsive if they are in opposite direc[tion]. One may note that the definition of the ampere is based on the force betwe[en] current carrying wires. Note that for long, parallel wires separated b[y] 1 meter with each carrying 1 ampere, the force per meter is

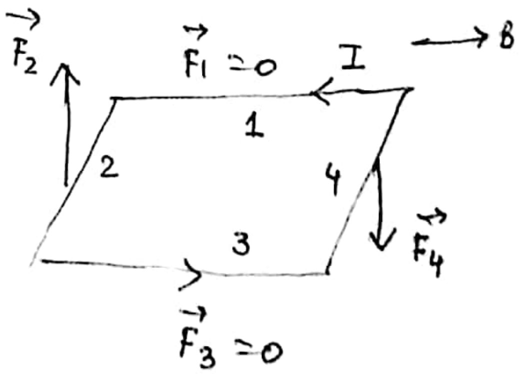
$$\frac{F}{l} = \frac{(4\pi \times 10^{-7}\ T.m/A).(1A)^2}{2\pi.(1m)} = 2 \times 10^{-7}\ N/m.$$

where, $\mu_0 = 4\pi \times 10^{-7}\ T.m/A$ and $1T = 1\ N/(A.m)$, the force per meter is $= 2 \times 10^{-7}\ N/m$. This is the basis of the definition of the ampere.

Torque on a current loop



A rectangular loop carrying a current I is in the xy plane. A magnetic field along the x-direction, $\vec{B} = B\hat{i}$ runs parallel to the plane of the loop, as shown in the figure alongside.



The magnetic forces acting on sides 1 and 3 vanishes because the length vectors $\vec{l_1} = -b\hat{i}$ and $\vec{l_3} = b\hat{i}$ are parallel and anti parallel to $\vec{B}$ and their cross-products vanish. The magnetic forces acting on segments 2 and 4 are non-vanishing.

$$\vec{F_2} = I(-a\hat{j}) \times (B\hat{i}) = IaB\hat{k}$$

$$\vec{F_4} = I(a\hat{j}) \times (B\hat{i}) = -IaB\hat{k}$$

with $\vec{F_2}$ pointing out of the page and $\vec{F_4}$ into the page. Thus the net force on the rectangular loop is –

$$\vec{F}_{net} = \vec{F_1} + \vec{F_2} + \vec{F_3} + \vec{F_4} = 0.$$

$$\therefore \boxed{\vec{F}_{net} = 0}$$

Even though the net force on the loop vanishes, the forces $\vec{F_2}$ and $\vec{F_4}$ will produce a torque which causes the loop to rotate about the y-axis (as shown in figure). The torque with respect to the centre of the loop is –

$$\vec{\tau} = \left(-\frac{b}{2}\hat{i}\right) \times \vec{F_2} + \left(\frac{b}{2}\hat{i}\right) \times \vec{F_4}$$

$$= \left(-\frac{b}{2}\hat{i}\right) \times (IaB\hat{k}) + \left(\frac{b}{2}\hat{i}\right) \times (-IaB\hat{k})$$

$$= \left(\frac{IabB}{2} + \frac{IabB}{2}\right)\hat{j}$$

$$= IabB\hat{j}$$

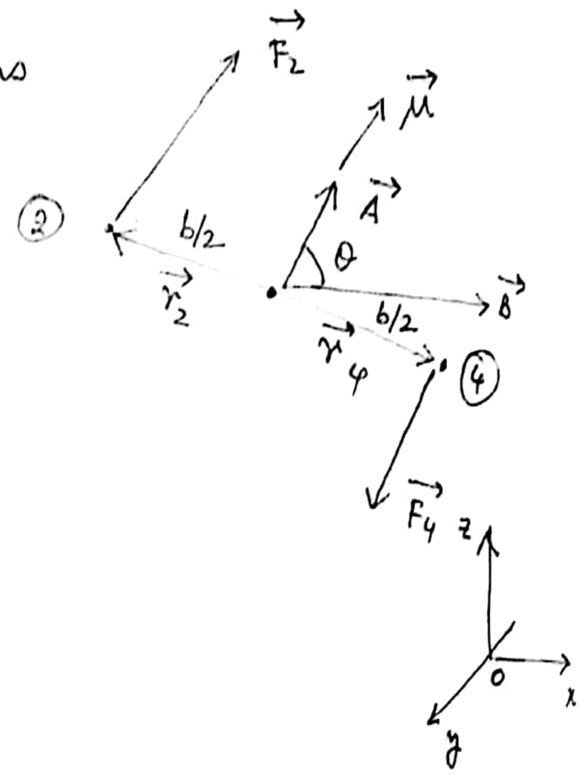
$$\therefore \boxed{\vec{\tau} = IAB\hat{j}}$$ where $A = ab$ represents the area of the loop and the positive sign indicates that the rotation is clockwise about the y-axis. It is convinient to introduce the area vector $\vec{A} = A\hat{n}$, where $\hat{n}$ is a unit vector in the direction normal to the plane of the loop. The direction of the positive sense of $\hat{n}$ is set by the conventional right hand rule. In our case, we have $\hat{n} = \hat{k}$. The above expression for torque can be rewritten as –

$$\boxed{\vec{\tau} = I\vec{A} \times \vec{B}}$$

The magnitude of the torque is at a maximum when $\vec{B}$ is parallel to the plane of the loop (or $\perp$r to $\vec{A}$).

Consider now the more general situation where the loop (or the area vector $\vec{A}$) makes an angle θ w.r.t. the magnetic field.

From the figure alongside, the lever arms can be expressed as –



$$\vec{r}_2 = \frac{b}{2}(\sin\theta\,\hat{i} + \cos\theta\,\hat{k})$$

$$= -\vec{r}_4$$

Hence, the net torque becomes –

$$\vec{\tau} = \vec{r}_2 \times \vec{F}_2 + \vec{r}_4 \times \vec{F}_4$$

$$= 2\,\vec{r}_2 \times \vec{F}_2$$

$$= 2 \cdot \frac{b}{2}(-\sin\theta\,\hat{i} + \cos\theta\,\hat{k}) \times (IaB\,\hat{k})$$

$$= IabB\sin\theta\,\hat{j}$$

or, $\boxed{\vec{\tau} = I\vec{A} \times \vec{B}}$

For a loop consisting of N turns, the magnitude of the torque is

$$\boxed{\tau = NIAB\sin\theta}.$$

Now, $\boxed{\vec{\mu} = NI\vec{A}}$ = magnetic dipole moment

The direction of $\vec{\mu}$ is the same as the area vector $\vec{A}$ ($\perp$r to the plane of the loop) and is determined by the right-hand rule. The SI unit for the magnetic dipole moment is $A \cdot m^2$. Using the expression for $\vec{\mu}$, the torque exerted on a current-carrying loop can be rewritten as -

$$\boxed{\vec{\tau} = \vec{\mu} \times \vec{B}}$$