Fermi Dirac statistics

Consider an ideal gas of N identical particles. Let s represent the state of a single particle and S the state of the whole gas. Suppose that when the gas is in state S, n_1 particles are in state $s=1$ with energy ϵ_1, n_2 particles in state $s=2$ with energy ϵ_2, etc. Then the total energy of the gas is – $E_S = \sum_s n_s \epsilon_s$. —(1).

and the total number of particles, $N = \sum_s n_s$ —(2).

The partition function can then be described by – $Z = \sum_S e^{-\beta E_S}$.

For Fermi Dirac system, the occupation number, n_s can be either 0 or 1. In the case of indistinguishable particle, the statistics can be treated using the grand canonical ensemble as N also is involved. The grand partition function is given by –

$$Z = \sum_S e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots + n_s\epsilon_s + \cdots) + \mu\beta(n_1 + n_2 + \cdots)}$$ —(3).

Since for F.D. statistics $n_s = 0$ or 1.

$$\therefore Z = (1 + e^{\beta(\mu-\epsilon_1)})(1 + e^{\beta(\mu-\epsilon_2)}) \cdots$$

$$\ln Z = \sum_s \ln(1 + e^{\beta(\mu-\epsilon_s)})$$

$$\therefore \bar{N} = \frac{1}{\beta}\frac{\partial \ln Z}{\partial \mu} = \sum_s \frac{e^{\beta(\mu-\epsilon_s)}}{1 + e^{\beta(\mu-\epsilon_s)}} = \sum_s \bar{n}_s$$ —(4).

$$\therefore \boxed{\bar{n}_s = \frac{1}{e^{\beta(\epsilon_s - \mu)} + 1}}$$ —(5)

This shows that if ϵ_s is very large $n_s \to 0$, if, on the other hand

ϵ_s is small, since the denominator is always greater than one, $\bar{n}_s << 1$.

The behaviour of the gas obeying FD statistics is different from that obeying BE statistics. This difference becomes particularly striking as T tends to zero, when the gas is in the state of lowest energy. In the case of BE statistics, since there is no restriction on the number of particles to be placed in any one single particle state, the gas will have the lowest energy when all the particles are in the lowest energy state. But in the F.D. statistics, since we can place only one particle in any one single particle state, even though the gas has lowest energy, one is forced to populate the successive states of higher energy with one particle in each. The lowest energy of a gas obeying FD statistics, therefore, is much higher than that it would have if the particles had obeyed BE statistics.

The Boltzmann limit of Boson and Fermion gases

We have seen that the mean occupation number of particles in an energy state ϵ_s is given by – $\bar{n}_s = \frac{1}{e^{\beta(\epsilon_s - \mu)} \pm 1}$

where the upper sign refers to FD statistics and lower to BE statistics. Suppose now $e^{-\beta\mu} >> 1$ i.e., $e^{\beta(\epsilon_s - \mu)} >> 1$. In this case –

$$\bar{n}_s \approx \frac{1}{e^{\beta(\epsilon_s - \mu)}} = e^{\beta\mu} e^{-\beta\epsilon_s}$$

Further because $\sum_s \bar{n}_s = N$, $e^{\beta\mu} \sum_s e^{-\beta\epsilon_s} = N$ or, $e^{\beta\mu} = \frac{N}{\sum_s e^{-\beta\epsilon_s}}$

and hence, $\bar{n}_s = \frac{N e^{-\beta\epsilon_s}}{\sum_s e^{-\beta\epsilon_s}}$, which is MB distribution.

Thus quantum statistics gives the Boltzmann form in the limiting case when $e^{-\beta\mu} >> 1$.

i.e., $\left(\frac{2\pi mkT}{h^2}\right)^{3/2}\left(\frac{V}{N}\right) \gg 1$.

This condition is satisfied if T is high and $\frac{N}{V} = \rho$ very low. Therefore, at high T, if $\frac{\epsilon-\mu}{kT}$ is to be high, μ must be negative and large in magnitude. Thus, for classical treatment to be valid

$$\frac{1}{\lambda^3}\left(\frac{V}{N}\right) \gg 1 \quad \text{or,} \quad \left(\frac{N\lambda^3}{V}\right) \ll 1$$

where we have introduced the mean thermal wavelength

$$\lambda = \frac{h}{(2\pi mkT)^{1/2}}.$$

A gas which can be treated classically is said to be non-degenerate, on the other hand if quantum statistical distribution are to be used, the gas is said to be degenerate.

Fermi energy

The Fermi Dirac distribution function which gives the average occupation of the energy level ϵ is given by -

$$f(\epsilon) = \bar{n}_\epsilon = \frac{1}{e^{\frac{\epsilon-\mu}{kT}} + 1} \quad \text{---(1)}.$$

Let us suppose that at $T=0$, the value of μ is μ_F. When $T>0$, $\bar{n}_\epsilon$ has two possible values -

$$\left.\begin{aligned} \bar{n}_\epsilon &= \frac{1}{e^{-\infty}+1} = 1 \quad \text{if } \epsilon < \mu_F. \\ \text{and,} \quad \bar{n}_\epsilon &= \frac{1}{e^{\infty}+1} = 0 \quad \text{if } \epsilon > \mu_F \end{aligned}\right\} \quad \text{---(2)}.$$

At absolute zero of temperature, the fermions will necessarily occupy the lowest available energy states. Immediate consequence of the Pauli exclusion principle is that each quantum state can contain only one fermion. Therefore, all the lowest states will be occupied with one fermion in each, until all the fermions are accomodated. Under this condition, the gas is said to be degenerate.

Expanding eqn. ①, near $\epsilon = \mu$, we get –

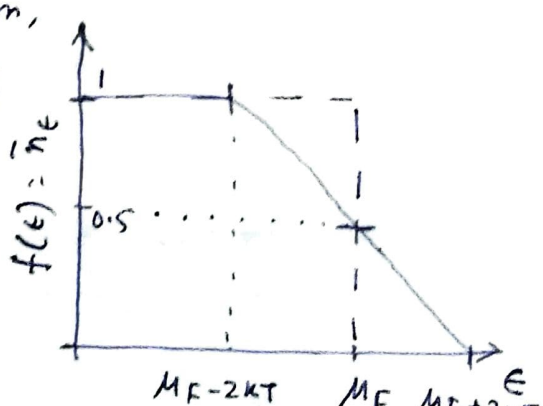
$$\bar{n}_\epsilon = \frac{1}{e^{\frac{\epsilon-\mu}{kT}}+1} = \frac{1}{2} - \frac{(\epsilon-\mu)}{4kT} + \ldots\ldots \quad \text{—③}.$$

$\therefore$ (i) if $\epsilon \leq \mu - 2kT$ $\quad \bar{n}_\epsilon = 1$ —④

(ii) if $\epsilon \geq \mu + 2kT$ $\quad \bar{n}_\epsilon = 0$ —⑤

(iii) if $\epsilon = \mu$ $\quad \bar{n}_\epsilon = \frac{1}{2}$ —⑥.

This distribution with varying energy at 0k is show in the fegivce alongside. The spread region of Fermi distribution, ie, the region of ϵ when $\bar{n}_\epsilon$ changes from unity to zero (from $\mu - 2kT$ to $\mu + 2kT$) narrows as the temperature decreases and at absolute zero becomes a sharp discontinuity. The distribution then, takes the form of a right angle. we see from this figure that at $T=0$, all states with $\epsilon < \mu_F$ are occupied; while those with $\epsilon > \mu_F$ are empty. The highest occupied level is called the Fermi level and is represented by ϵ_F. Below this level there are exactly N states, if N is the total number of fermions. The states above this energy level are unoccupied. The value of ϵ_F depend



upon the number of fermions in the gas. One can easily see that at $T=0$, ϵ_F coincides with μ_F.

The value of ϵ_F can be found as follows: $\int_0^\infty n(\epsilon)\,d\epsilon = N$ —— (7)

where N is the total number of particles in the gas. Since at $T=0$ all single particle states upto ϵ_F are completely filled with one particle per state, for $\epsilon < \epsilon_F$, the number of states $\sigma(\epsilon) = n(\epsilon)$ and for $\epsilon > \epsilon_F$ the number of states $\sigma(\epsilon) = 0$.

$$\therefore \int_0^{\epsilon_F} \sigma(\epsilon)\,d\epsilon = N \qquad \text{—— (8)}.$$

If the particles have spin s, there are $2s+1$ single particle states all having the same energy ϵ. Therefore, the density of states $\sigma(\epsilon)\,d\epsilon$ in the energy range ϵ to $\epsilon+d\epsilon$ is given by -

$$\sigma(\epsilon)\,d\epsilon = \frac{V \cdot 2\pi \cdot (2m)^{3/2}\,\epsilon^{1/2}\,(2s+1)}{h^3}\,d\epsilon \qquad \text{—— (9)}$$

Hence equation (9) can be written as -

$$\int_0^{\epsilon_F} \frac{N(2s+1)\,2\pi\,(2m)^{3/2}}{h^3}\,\epsilon^{1/2}\,d\epsilon = N \qquad \text{—— (10)}.$$

i.e., $$\frac{2\pi(2s+1)V}{h^3} \cdot (2m)^{3/2} \cdot \frac{2}{3}\,\epsilon_F^{3/2} = N \qquad \text{—— (11)}.$$

or, $$\epsilon_F = \frac{h^2}{2m}\left[\frac{3N}{4\pi(2s+1)V}\right]^{2/3} \qquad \text{—— (12)}.$$

The value of the single particle momentum corresponding to the Fermi energy is referred to as Fermi momentum and is denoted by p_F.

$$p_F = \sqrt{2m\epsilon_F} = \hbar\left(\frac{3N}{4\pi(2s+1)V}\right)^{1/2} \quad \text{— (13)}$$

Mean energy of fermions at T > 0K.

$$\bar{\epsilon}(0) = \frac{\int_0^\infty \epsilon\, n(\epsilon)\, d\epsilon}{\int_0^\infty n(\epsilon)\, d\epsilon} \quad \text{— (14)}.$$

$$= \frac{\int_0^{\epsilon_F} \epsilon\, g(\epsilon)\, d\epsilon}{\int_0^{\epsilon_F} g(\epsilon)\, d\epsilon} = \frac{2\pi V(2s+1)\left(\frac{2m}{h^2}\right)^{3/2}\int_0^{\epsilon_F}\epsilon^{3/2}\, d\epsilon}{2\pi V(2s+1)\left(\frac{2m}{h^2}\right)^{3/2}\int_0^{\epsilon_F}\epsilon^{1/2}\, d\epsilon}$$

$$\therefore \bar{\epsilon}(0) = \frac{\int_0^{\epsilon_F}\epsilon^{3/2}\, d\epsilon}{\int_0^{\epsilon_F}\epsilon^{1/2}\, d\epsilon} = \frac{3}{5}\epsilon_F \quad \text{— (15)}.$$

The ground state, or zero point energy, therefore, is $-E_0 = \frac{3}{5}N\epsilon_F$ — (16).

Substituting N from eqn. (12), $E_0 = \frac{4\pi(2s+1)V}{5}\left(\frac{2m}{h^2}\right)^{3/2}\epsilon_F^{5/2}$ — (17).

The ground state pressure of the system is, $P_0 = \frac{2}{3}\left(\frac{E_0}{V}\right) = \frac{2}{5}\frac{N}{V}\epsilon_F$

or, $P_0 = \frac{2}{5}n\epsilon_F$ — (18). where $n = N/V$.

Fermi energy as a function of temperature –

For a FD system, to find the variation of Fermi energy at temperatures other than 0K, we need to evaluate the Fermi function $f(\epsilon)$, i.e., integrals of the type $\int_0^\infty f(\epsilon)\varphi(\epsilon)\, d\epsilon$. — (19)

where the Fermi function $f(\epsilon) = \frac{1}{e^{\frac{\epsilon-\mu}{KT}}+1}$ and $\varphi(\epsilon)$ a smoothly varying function of ϵ.

Integrating by parts, $I = \left[f(\epsilon)\int_0^{\epsilon}\varphi(\epsilon)\,d\epsilon\right]_0^{\infty} - \int_0^{\infty}\psi(\epsilon)\,f'(\epsilon)\,d\epsilon$ — (20).

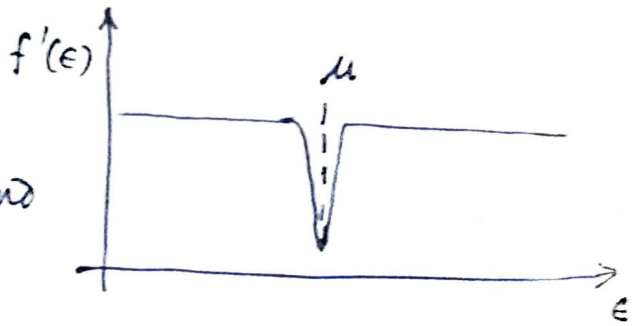
where $\psi(\epsilon) = \int_0^{\epsilon}\varphi(\epsilon)\,d\epsilon$.

Now, $f(\infty) = \frac{1}{e^{\infty}+1} = 0$, and, $\psi(0) = \int_0^{0}\varphi(\epsilon)\,d\epsilon = 0$.

Therefore, the first term in (20) vanishes at both the limits.

Hence, $I = -\int_0^{\infty}\psi(\epsilon)\,f'(\epsilon)\,d\epsilon$ — (21).

Since, in all practical cases, the temperature is small compared with the Fermi temperature of the gas, the differential $f'(\epsilon)$ will be negligible except in the neighbourhood of the Fermi energy ϵ_F. The form of $f'(\epsilon)$ as a function of ϵ is shown in the figure alongside.



only relatively narrow range about $\epsilon = \epsilon_F$ contributes appreciably to the integrals. Let us, therefore, expand $\psi(\epsilon)$ about ϵ_F in a power series

$$\psi(\epsilon) = \psi(\epsilon_F) + \psi'(\epsilon_F)(\epsilon-\epsilon_F) + \psi''(\epsilon_F)\frac{(\epsilon-\epsilon_F)^2}{2!} + \ldots\ldots \quad — (22).$$

$$\therefore I = -\int_0^{\infty}\psi(\epsilon_F)\,f'(\epsilon)\,d\epsilon - \int_0^{\infty}\psi'(\epsilon_F).(\epsilon-\epsilon_F)f'(\epsilon)\,d\epsilon$$

$$- \int_0^{\infty}\psi''(\epsilon_F).\frac{(\epsilon-\epsilon_F)^2}{2}f'(\epsilon)\,d\epsilon \quad — (23).$$

$$= -I_1 - I_2 - I_3 \text{ neglecting the higher terms}$$

Now, $I_1 = \int_0^{\infty}\psi(\epsilon_F)\,f'(\epsilon)\,d\epsilon = \psi(\epsilon_F)\int_0^{\infty}f'(\epsilon)\,d\epsilon = \psi(\epsilon_F)\left[f(\epsilon)\right]_0^{\infty}$

$$= -\psi(\epsilon_F) \cdot \frac{1}{e^{-\epsilon_F/kT}+1}$$

and because $\epsilon_F \gg kT$, $I_1 = -\psi(\epsilon_F) = -\int_0^{\epsilon_F} \varphi(\epsilon)\, d\epsilon$ — (24).

$$\therefore I_2 = \int_0^{\infty} \psi'(\epsilon_F)\,(\epsilon-\epsilon_F)\, f'(\epsilon)\, d\epsilon$$

$$= -\psi'(\epsilon_F) \int_0^{\infty} \frac{(\epsilon-\epsilon_F)\, e^{\frac{(\epsilon-\epsilon_F)}{kT}}}{kT\left[e^{\frac{\epsilon-\epsilon_F}{kT}}+1\right]^2} d\epsilon \quad \text{— (25)}.$$

putting $x = \frac{\epsilon-\epsilon_F}{kT}$, $I_2 = \psi'(\epsilon_F)\, kT \int_{-\frac{\epsilon_F}{kT}}^{\infty} \frac{x e^x}{(e^x+1)^2} dx$ — (26).

The integral being an odd function, the definite integral changes sign if x is replaced by $-x$. Therefore, the integral is zero.

Hence, $I_2 = 0$ — (27).

$$I_3 = \int_0^{\infty} \psi''(\epsilon_F) \cdot \frac{(\epsilon-\epsilon_F)^2}{2} f'(\epsilon)\, d\epsilon$$

$$= -\frac{\psi''(\epsilon_F)}{2} (kT)^2 \int_{-\infty}^{\infty} \frac{x^2 e^x}{(e^x+1)^2} dx. \quad \text{— (28)}$$

In this case, the integral is an even function of x

$$\therefore I_3 = -\psi''(\epsilon_F)(kT)^2 \int_0^{\infty} \frac{x^2 e^x}{(e^x+1)^2} dx$$

$$= -\psi''(\epsilon_F)(kT)^2 \int_0^{\infty} \frac{x^2 e^{-x}}{(e^{-x}+1)^2} dx. \quad \text{— (29)}.$$

Now $(1+e^{-x})^{-2} = 1 - 2e^{-x} + 3e^{-2x} - 4e^{-3x} + \ldots$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} n\, e^{-(n-1)x} \quad \text{— (30)}.$$

$$\therefore \int_0^\infty \frac{x^2 e^{-x}}{(1+e^{-x})^2}\,dx = \int_0^\infty \sum_{n=1}^{\infty} (-1)^{n-1} n\, x^2 e^{-nx}\,dx$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} n \int_0^\infty x^2 e^{-nx}\,dx$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} n \cdot \frac{2}{n^3} = 2\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \text{— (31)}.$$

$$\therefore I_3 = -\psi''(\epsilon_F)(kT)^2 \frac{\pi^2}{6} = -\varphi'(\epsilon_F)(kT)^2 \frac{\pi^2}{6} \qquad \text{— (32)}.$$

(∵ From $\psi(\epsilon) = \int_0^\epsilon \varphi(\epsilon)\,d\epsilon$, $\psi'(\epsilon) = \varphi(\epsilon)$ and $\psi''(\epsilon) = \varphi'(\epsilon)$)

Hence, $I = -I_1 - I_3$

$$= \int_0^{\epsilon_F} \varphi(\epsilon)\,d\epsilon + \varphi'(\epsilon_F)(kT)^2 \cdot \frac{\pi^2}{6} \qquad \text{— (33)}.$$

we know that, $N = \int_0^\infty f(\epsilon)\sigma(\epsilon)\,d\epsilon$ — (34).

using eqn. (33), and replacing $\varphi(\epsilon)$ by $\sigma(\epsilon)$ we get

$$N = \int_0^{\epsilon_F} \sigma(\epsilon)\,d\epsilon + \sigma'(\epsilon_F)(kT)^2 \frac{\pi^2}{6} \qquad \text{— (35)}.$$

Since $N = \int_0^{\epsilon_F(0)} \sigma(\epsilon)\,d\epsilon$ — (36).

where $\epsilon_F(0)$ is the Fermi energy at $T = 0K$.

$$\therefore \sigma'(\epsilon_F)(kT)^2 \frac{\pi^2}{6} = \int_0^{\epsilon_F(0)} \sigma(\epsilon)\,d\epsilon - \int_0^{\epsilon_F} \sigma(\epsilon)\,d\epsilon$$

$$= \int_{\epsilon_F}^{\epsilon_F(0)} \sigma(\epsilon)\,d\epsilon \qquad \text{— (37)}.$$

Since the Fermi energy ϵ_F differs only slightly from its value $\epsilon_F(0)$ at 0K, we may write –

$$\int_{\epsilon_F}^{\epsilon_F(0)} \sigma(\epsilon)\,d\epsilon = \sigma(\epsilon_F(0))\,(\epsilon_F(0) - \epsilon_F) \quad \text{—(38)}.$$

$$\therefore \sigma'(\epsilon_F)\,(kT)^2\,\frac{\pi^2}{6} = \sigma\{\epsilon_F(0)\}\,(\epsilon_F(0) - \epsilon_F) \quad \text{—(39)}.$$

Since, $\sigma(\epsilon) = \frac{4\pi V}{h^3}\cdot(2m)^{3/2}\,\epsilon^{1/2}$

$$\therefore \sigma'(\epsilon) = \frac{4\pi V}{h^3}\cdot\frac{(2m)^{3/2}}{2\epsilon^{1/2}} = \frac{\sigma(\epsilon)}{2\epsilon} \quad \text{—(40)}.$$

$\therefore$ eqn. (39) changes to, $\frac{\sigma(\epsilon_F)}{2\epsilon_F}(kT)^2\cdot\frac{\pi^2}{6} = \sigma(\epsilon_F(0))(\epsilon_F(0) - \epsilon_F)$ —(41).

Further, because $\epsilon_F \simeq \epsilon_F(0)$

$$\frac{\sigma(\epsilon_F)}{\epsilon_F} = \frac{\sigma(\epsilon_F(0))}{\epsilon_F(0)}$$

and therefore $\frac{\sigma(\epsilon_F(0))}{2\epsilon_F(0)}(kT)^2\,\frac{\pi^2}{6} = \sigma(\epsilon_F(0))(\epsilon_F(0) - \epsilon_F)$ —(42).

ie, $\frac{(kT)^2}{\epsilon_F(0)}\cdot\frac{\pi^2}{12} = \epsilon_F(0) - \epsilon_F$ —(43).

Hence, $\epsilon_F = \epsilon_F(0)\left[1 - \frac{(kT)^2}{(\epsilon_F(0))^2}\cdot\frac{\pi^2}{12}\right]$ —(44).

$$\boxed{\epsilon_F = \epsilon_F(0)\left\{1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right\}} \quad \text{—(45)}$$

Free electron theory of metal (electron gas) —

Drude in 1900 postulated that the metals consist of positive ion cores with the valence electrons moving freely among these cores. The electrons are, however, bound to move within the metal due to electrostatic attraction between the positive ion cores and the electrons. The potential field of these ion cores, which is responsible for such an interaction is assumed to be constant throughout the metal and the mutual repulsion among the electrons is neglected. The behaviour of free electrons moving inside the metals is considered to be similar to that of atoms or molecules in perfect gas. These free electrons are, therefore, also referred to as free electron gas and the theory is accordingly named as free electron gas model.

The free electron gas, however, differs from an ordinary gas in some respects. Firstly the free electron gas is negatively charged whereas the molecules of an ordinary gas are mostly neutral. Secondly, the concentration of electrons in an electron gas is quite large as compared to the concentration of molecules in an ordinary gas. These electrons are responsible for conduction of electricity through metals.

By applying the methods used in the kinetic theory of gases Drude was able to give a quantitative derivation of Wiedemann-Franz law, that the ratio of thermal to electrical conductivities is a universal number times the absolute temperature. The calculated Lorentz number was in reasonable agreement with experiment.

The main drawbacks of Drude's theory are -

(i) The observed specific heat of metals - 3R at room temperature, appears to be completely accountable by the lattice vibrations alone; while the theory demands that on the basis of the equi partition theorem, every electron must make a contribution of magnitude $\frac{3}{2}k$ to the specific heat so that the total heat capacity should be $\frac{9}{2}R$.

(ii) Failure to explain magnetic properties, and

(iii) Failure to explain the temperature dependence of the conductivity.

The first difficulty was resolved by Sommerfield by the application of FD statistics to the electron gas in metals. Pauli and Heisenberg showed how quantum theory can be used to explain the magnetic properties; and the concepts necessary to solve the third problem were provided by Sommerfild.

Considering the free nature of valence ~~electrons~~ electrons as assumed in the classical theory, Sommerfild treated the problem quantum mechanically using the Fermi Dirac statistics. The possible electronic energy states in the potential energy box and the distribution of electrons in these states are then determined using quantum statistics.

Considering an one dimensional potential box $V(x) = 0$ for $0 < x < L$
$= \infty$ for $x \leq 0$ and $x \geq L$

the wavefunction ψ_n of the electron occupying the n^{th} state is determined from the solution of Schrodinger equation, ie,

$$\frac{d^2\psi_n}{dx^2} + \frac{2m}{\hbar^2}(E_n - V)\psi_n = 0 ,$$

from which the normalized $\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$ and the energy eigenvalue $E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$.

Fermi energy of electron gas in metal

The electrons are distributed among the various possible energy levels in accordance with Pauli's exclusion principle which states that no two electrons can have all their quantum number identical. In a solid, an electron in a conduction electronic state has the quantum no. n and m_s where n is the principle quantum number and m_s is the magnetic spin quantum number. For each value of n, m_s can have two possible values $+1/2$ and $-1/2$. So, each energy level is doubly degenerate. The topmost filled energy level at 0K is known as Fermi level and the energy corresponding to this level is called the Fermi energy E_F.

If N is the total number of electrons to be accomodated on the energy level then, for even n, $2n_F = N$, where n_F is the principle quantum number of the Fermi level. Thus for $n = n_F$, we can write

$$E_F = \frac{\hbar^2}{2m}\left(\frac{\pi^2 n_F^2}{L^2}\right) = \frac{\hbar^2}{2m}\left(\frac{N\pi}{2L}\right)^2$$

The value of the Fermi energy depends on the length of the box and the number of electrons in the box.

The total energy E_0 of all the N electrons in the ground state is given by - $E_0 = 2\sum_{n=1}^{N/2} E_n$. The factor 2 appears since each level can accomodate two electrons of up and down spin.

$$\therefore E_0 = 2\frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 \sum_{n=1}^{N/2} n^2 \qquad \left(\because E_n = \frac{\hbar^2}{2m}\cdot\left(\frac{n\pi}{L}\right)^2\right)$$

Since $\sum_{n=1}^{S} n^2 = \frac{1}{6}S(2S^2+3S+1) \simeq \frac{1}{3}S^3$ for $S \geq 1$.

$$\therefore \sum_{n=1}^{N/2} n^2 = \frac{1}{3}\left(\frac{N}{2}\right)^3.$$

$$\therefore E_0 = 2\cdot\frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 \frac{1}{3}\left(\frac{N}{2}\right)^3 = \frac{1}{3}\frac{\hbar^2}{2m}\left(\frac{N\pi}{2L}\right)^2 N = \frac{1}{3}NE_F.$$

$$\left(\because E_F = \frac{\hbar^2}{2m}\left(\frac{N\pi}{2L}\right)^2\right)$$

Density of states

The density of states is defined as the number of electronic states present in a unit energy range — $D(E) = 2\frac{dn}{dE}$.

$$\therefore \frac{dE}{dn} = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 \cdot 2n = \frac{h^2 n}{4mL^2} \qquad \left(\because E = \frac{\hbar^2}{2m}\frac{n^2\pi^2}{L^2}\right).$$

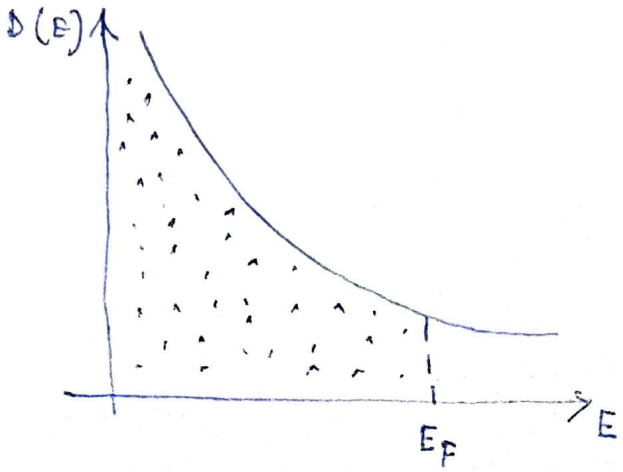
$$\therefore D(E) = 2\left(\frac{4mL^2}{nh^2}\right) = \frac{8mL^2}{h^2}\cdot\frac{1}{n}$$

Again, $\frac{1}{n} = \left(\frac{h^2}{8mL^2E}\right)^{1/2}$ $\left(\because \text{from } E = \frac{\hbar^2}{2m}\cdot\frac{n^2\pi^2}{L^2}\right)$

and $\hbar = h/2\pi$

Hence, $D(E) = \left(\frac{8mL^2}{h^2E}\right)^{1/2}$

$$= \frac{4L}{h}\left(\frac{m}{2E}\right)^{1/2}.$$



The plot of $D(E)$ vs. E indicates that all the levels present below the Fermi level are filled and all those present above it are empty at OK.

Free electron gas in three dimensions

The potential of the electrons inside the crystal is constant and may be taken as zero, whereas it has a large value outside the crystal. For simplicity, the three dimensional crystal may be regarded as a cubical box having length L. The free particle Schrodinger eqn. in 3D is –

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\psi_k(\vec{r}) = E_k\,\psi_k(\vec{r}).$$

where E_k is the energy of the electron in k – state. The solution to this equation is –

$$\psi_k(\vec{r}) = \sqrt{\frac{8}{L^3}}\sin\left(\frac{n_x\pi x}{L}\right)\sin\left(\frac{n_y\pi y}{L}\right)\sin\left(\frac{n_z\pi z}{L}\right)$$

where n_x, n_y and n_z are positive integers and $\sqrt{8/L^3}$ is the normalizing factor. The corresponding allowed energy eigenvalues are,

$$E_k = \frac{\hbar^2k^2}{2m} = \frac{\hbar^2}{2m}\left(k_x^2+k_y^2+k_z^2\right) \qquad \left(k=\frac{2\pi}{\lambda}\right).$$

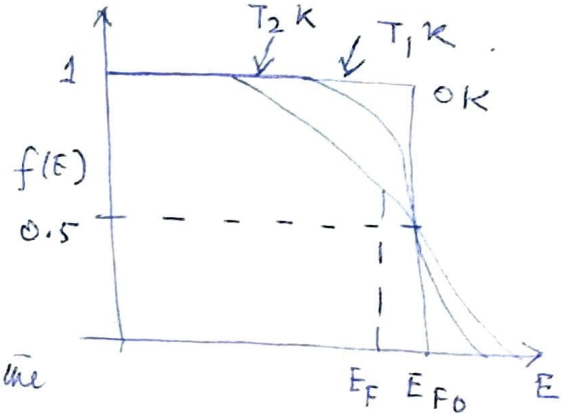
The distribution of electrons in the allowed energy levels follows the Pauli exclusion principle. A total of N non interacting electrons at 0K can be filled in N/2 energy levels. The topmost filled level is the (N/2) th level and all levels lying above it are empty. This level is therefore the Fermi level as it divides the filled and empty level at 0K. The energy of the Fermi level, or the Fermi energy is denoted by E_{F0}. However, for temperatures greater than 0K, the Fermi energy level may not be the topmost filled level since some of the electrons from the filled energy levels may be excited to higher levels. Thus some of the levels below E_{F0} would be empty while some above it would be occupied. The probability that a particular quantum state of energy E is occupied

at a temperature T is given by the so called Fermi function –

$$f(E) = \frac{1}{\exp\left(\frac{E-E_F}{kT}\right)+1} \quad \text{— (1)},$$

where E_F is the Fermi energy.

The plot of $f(E)$ vs. E for different temperatures is shown below.



At absolute zero, $f(E) = 1$ for $E \leq E_{FO}$

$= 0$ for $E > E_{FO}$

For temperatures greater than 0K but less than the melting point of the metal such that $k_B T << E_F$, the distribution function loses the step character. The probability of occupation $f(E)$, decreases gradually from 1 to 0 near E_F. This indicate that some of the states below E_F are empty while some others above it are filled. This is because some of the electrons from the energy states below E_F gain thermal energy and get excited to the states above E_F. At $E = E_F$, eqn. (1) gives – $f(E_F) = \frac{1}{2}$. Thus for temperatures greater than 0K, the Fermi level may be defined as the level where the probability of occupation is 1/2. Unlike E_{FO} it is not the topmost filled level; instead, it lies between the filled level and empty levels. The position of the Fermi level is not fixed, but changes with temperature. An approximate relationship between E_F and E_{FO} is given by –

$$E_F \simeq E_{FO}\left[1 - \frac{\pi^2}{12}\left(\frac{kT}{E_{FO}}\right)^2\right] \quad \text{— (2)}.$$

For energies below E_F such that $(E_F - E) \gg kT$, $f(E) \simeq 1$, provided T does not differ much from 0K. Thus it is only in the vicinity of E_F minus a few kT that $f(E)$ becomes less than unity. It, therefore,

follows that as the temperature is raised above 0K, all the electrons would not gain energy as expected classically, but only those which lie within an energy range KT below the Fermi level can do so. This is because the electrons present near the Fermi level can jump to the vacant higher energy states after acquiring energy of the order of KT, but the electrons present well below the Fermi level cannot do so due to non availability of empty states within the range KT. Thus according to quantum mechanics, only a fraction of the electrons can gain thermal energy and get excited to the energy states. The value of this fraction is $KT/E_F \simeq 0.01$ at room temperature for $E_F = 3.0$ eV. Note that according to the classical mechanics, KT represents the energy acquired by a particle at a temperature T K. At room temperature $KT \simeq 0.025$ eV.

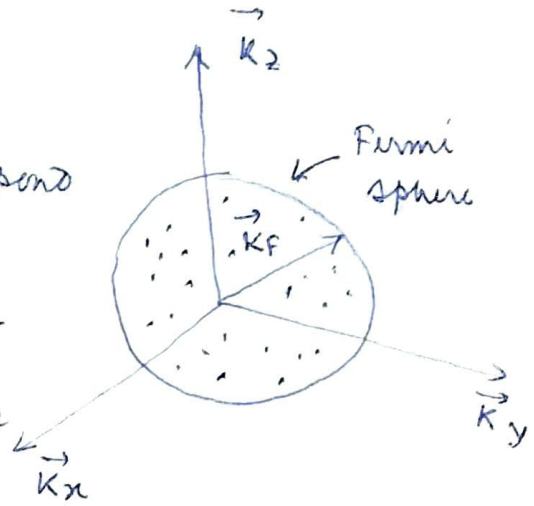
Density of available electron states, D(E)

To obtain the expression of D(E), we consider the linear momentum $\vec{p}$ which, in quantum mechanics is represented by the operator $\hat{p} = -i\hbar\nabla$ such that, $\hat{p}\,\psi_k(\vec{r}) = -i\hbar\nabla\psi_k(\vec{r}) = \hbar k\,\psi_k(\vec{r})$ —(3).

This indicates that the plane wave function ψ_k is an eigenfunction of the linear momentum with the eigenvalue $\hbar k$. The particle velocity is given by - $v = \hbar k/m$ —(4).

In a system of N electrons, the occupied states or orbitals in the ground state may be represented by points inside a sphere in k-space. The energy corresponding to the surface of the sphere then represents the Fermi energy. Let $\vec{k_F}$ be the wave vector from the origin of the $\vec{k}$-space

to the surface of the sphere as shown in figure alongside. Then, the Fermi energy is — $E_F = \frac{\hbar^2 k_F^2}{2m}$ — (5).



There is only one allowed wave vector or one distinct triplet of (k_x, k_y, k_z) which corresponds to the volume element $(2\pi/L)^3$ of $\vec{k}$-space. Thus, in the sphere called the Fermi sphere of volume $\frac{4\pi}{3} k_F^3$, the total number of electronic states or orbitals is —

$$2 \cdot \frac{\frac{4}{3}\pi k_F^3}{\left(\frac{2\pi}{L}\right)^3} = \frac{L^3}{3\pi^2} \cdot k_F^3 = \frac{V}{3\pi^2} k_F^3 = N. \quad \text{— (6)}.$$

$$\therefore \quad k_F = \left(\frac{3\pi^2 N}{V}\right)^{1/3} \quad \text{— (7)}.$$

and $$E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3} \quad \text{— (8)}.$$

The total number of electrons is, $N = \frac{V}{3\pi^2}\left(\frac{2mE_F}{\hbar^2}\right)^{3/2}$ — (9).

and velocity v_F at the Fermi surface is — $v_F = \frac{\hbar k_F}{m} = \frac{\hbar}{m}\left(\frac{3\pi^2 N}{V}\right)^{1/3}$

The density of states, $D(E)$ is now obtained as — (10).

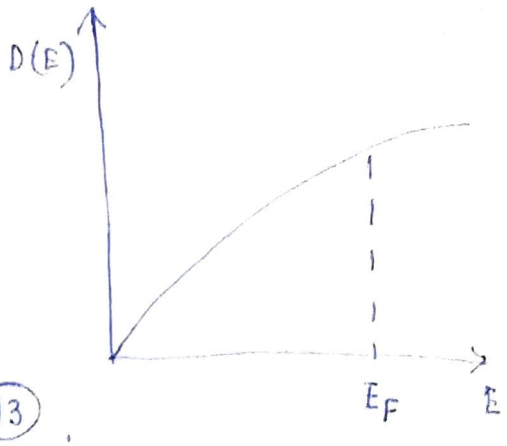
$$\int_0^{E_F} D(E)\, dE = N. \quad \text{— (11)}.$$

$$\therefore \quad \int_0^{E_F} D(E)\, dE = \frac{V}{3\pi^2}\left(\frac{2mE_F}{\hbar^2}\right)^{3/2}.$$

or, $$\int D(E)\, dE = \frac{V}{3\pi^2}\left(\frac{2mE}{\hbar^2}\right)^{3/2} = \int \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2}\, dE.$$

or, $$D(E) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2} \quad \text{— (12)}.$$

The variation of $D(E)$ with E is parabolic as shown in the adjacent figure. At the Fermi surface, the density of states is obtained by putting $E = E_F$, i.e.,



$$D(E_F) \equiv \frac{dN}{dE_F} = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E_F^{1/2} \quad \text{—(13)}$$

The Fermi energy, E_{FO} –

The total number of electrons can be obtained by integrating

$$N = \int_0^\infty N(E)\,dE = \int_0^\infty D(E)\,f(E)\,dE \quad \text{—(14)}.$$

$$\therefore \quad N = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty E^{1/2} f(E)\,dE$$

$$= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \left[\int_0^{E_{FO}} E^{1/2} f(E)\,dE + \int_{E_{FO}}^\infty E^{1/2} f(E)\,dE\right]$$

$$\therefore \quad N = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^{E_{FO}} E^{1/2}\,dE \qquad \left(\because f(E) = 1,\ E \le E_{FO};\ = 0,\ E > E_{FO}\right)$$

$$= \frac{V}{3\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E_{FO}^{3/2}$$

or, $E_{FO} = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$. Putting $n = N/V$, the electron concentration, we get, $E_{FO} = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3} = \frac{\hbar^2}{8m}\left(\frac{3n}{\pi}\right)^{2/3}$.

Average kinetic energy, $\bar{E}_0$

$$\bar{E}_0 = \frac{1}{N}\int_0^\infty E\,D(E)\,f(E)\,dE$$

$$\therefore \quad \bar{E}_0 = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \cdot \frac{1}{N}\left[\int_0^{E_{FO}} E \cdot E^{1/2} f(E)\,dE + \int_{E_{FO}}^\infty E \cdot E^{1/2} f(E)\,dE\right]$$

$$\therefore \quad \bar{E}_0 = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \cdot \frac{1}{N} \int_0^{E_{F0}} E^{3/2}\, dE$$

$$= \frac{2}{5} \cdot \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \cdot \frac{1}{N} E_{F0}^{5/2}$$

Substituting the value of N from eqn. (9), we obtain –

$$\bar{E}_0 = \frac{3}{5} E_{F0} .$$

Application of free electron gas model (electronic specific heat) –

The average kinetic energy of a free electron as given by the classical statistical mechanics is – $\bar{E}_0 = \frac{3}{2} kT$, For N free electrons, $E = N\bar{E} = \frac{3}{2} NkT$ and, $C_v = \left.\frac{\partial E}{\partial T}\right|_v = \frac{3}{2} Nk$. — (i)

However, at room temperature, the electronic contribution to specific heat is not more than 0·01 of the value of eqn. (i). Also, this contribution decreases linearly to zero as $T \to 0$. This can be explained as follows – The quantum mechanics suggests that only those electrons contribute to the specific heat which lie within an energy range kT below the Fermi level. This happens because when the electron gas is heated up to a temperature T, these electrons acquire energy of the order of kT and jump to the empty higher energy states. The deep lying electrons cannot do so because the unfulfilled energy states are not available to these electrons for excitation. These electrons as such, do not contribute to the specific heat. The number of electrons which contribute to the specific heat is of the order $N\left(\frac{kT}{E_F}\right)$ or $N\left(\frac{T}{T_F}\right)$ where T_F is the Fermi temperature, given by – $E_F = kT_F$.

$\therefore$ Eqn. ① becomes, $C_v \simeq \frac{3}{2} Nk \left(\frac{T}{T_F}\right)$ —②.

This indicates that C_v is proportional to T and approaches zero at $T \to 0$. Using typical values of T and T_F as 300K and 30000K respectively, the eqn. ② gives, $C_v \simeq 0.01 \cdot \frac{3}{2} Nk_B$ —③.

The specific heat calculated from this eqn. agrees with the experimental value.

The electronic specific heat (in metal) —

The total energy E is given by, $E = \int_0^\infty \epsilon f(\epsilon) \sigma(\epsilon) d\epsilon$. —①

where $f(\epsilon)$ gives the average occupation of the energy level ϵ given by — $f(\epsilon) = \bar{n}_\epsilon = \frac{1}{\exp\left(\frac{\epsilon - \epsilon_F}{kT}\right) + 1}$ and, for $\epsilon < \epsilon_F$, the number of states $\sigma(\epsilon) = n(\epsilon)$ and for $\epsilon > \epsilon_F$ the number of states $\sigma(\epsilon) = 0$.

$$E = \int_0^{\epsilon_F} \epsilon \sigma(\epsilon) d\epsilon + \sigma(\epsilon_F)(kT)^2 \frac{\pi^2}{6} + \epsilon_F \sigma'(\epsilon_F)(kT)^2 \frac{\pi^2}{6} \quad —②$$

Now, $\int_0^{\epsilon_F} \epsilon \sigma(\epsilon) d\epsilon = \int_0^{\epsilon_F(0)} \epsilon \sigma(\epsilon) d\epsilon - \int_{\epsilon_F}^{\epsilon_F(0)} \epsilon \sigma(\epsilon) d\epsilon$

$$= E_0 - \epsilon_F(0) \sigma(\epsilon_F(0)) (\epsilon_F(0) - \epsilon_F) \quad —③.$$

where $E_0 = \int_0^{\epsilon_F(0)} \epsilon \sigma(\epsilon) d\epsilon$ is the total energy of the electrons at $T = 0K$.

But, $\epsilon_F(0) = \epsilon_F + \frac{(kT)^2}{\epsilon_F(0)} \cdot \frac{\pi^2}{12}$

$$\therefore \int_0^{\epsilon_F} \epsilon\, \sigma(\epsilon)\, d\epsilon = E_0 - \sigma(\epsilon_F(0))(kT)^2 \cdot \frac{\pi^2}{12} \quad \text{—(4)}$$

The last two terms in equation (2) can be expressed as —

$$\left\{\sigma(\epsilon_F) + \epsilon_F \sigma'(\epsilon_F)\right\}(kT)^2 \cdot \frac{\pi^2}{6} = \left\{\sigma(\epsilon_F) + \frac{\sigma(\epsilon_F)}{2}\right\}(kT)^2 \frac{\pi^2}{6}$$

$$= \frac{3}{2}\sigma(\epsilon_F)(kT)^2 \frac{\pi^2}{6} \approx \frac{3}{2}\sigma(\epsilon_F(0))(kT)^2 \cdot \frac{\pi^2}{6} \quad \text{—(5)}$$

$$\therefore E = E_0 - \sigma(\epsilon_F(0))(kT)^2 \cdot \frac{\pi^2}{12} + \frac{3}{2}\sigma(\epsilon_F(0))(kT)^2 \cdot \frac{\pi^2}{6}$$

$$= E_0 + \sigma(\epsilon_F(0))(kT)^2 \cdot \frac{\pi^2}{6} \quad \text{—(6)}$$

Now, $E_0 = \frac{3}{5} N \epsilon_F(0)$ and $N = \int_0^{\epsilon_F(0)} \sigma(\epsilon)\, d\epsilon$

or, $N = \frac{4\pi V}{h^3} \cdot (2m)^{3/2} \int_0^{\epsilon_F(0)} \epsilon^{1/2}\, d\epsilon$

$$= \frac{2}{3} \cdot \frac{4\pi V}{h^3} (2m)^{3/2} [\epsilon_F(0)]^{3/2}$$

$$= \frac{2}{3} \epsilon_F(0)\, \sigma(\epsilon_F(0)) \quad \text{—(7)}$$

or, $\sigma(\epsilon_F(0)) = \frac{3N}{2\epsilon_F(0)} \quad \text{—(8)}$

Hence, $E = \frac{3}{5} N \epsilon_F(0) + \frac{3N}{2\epsilon_F(0)} (kT)^2 \frac{\pi^2}{6}$

$$= N\left\{\frac{3}{5}\epsilon_F(0) + \frac{k^2 T^2 \pi^2}{4\epsilon_F(0)}\right\} \quad \text{—(9)}$$

$\therefore$ Mean energy of an electron $\bar{\epsilon} = \frac{E}{N} = \frac{3}{5}\epsilon_F(0) + \frac{(kT)^2 \pi^2}{4\epsilon_F(0)} \quad \text{—(10)}$

$$\simeq \epsilon_F(0)\left\{\frac{3}{5}+\frac{\pi^2}{4}\cdot\frac{T^2}{T_F^2}\right\} \quad -(11)$$

The electronic specific heat C_v is — $C_v = \left(\frac{\partial E}{\partial T}\right)_V$

$$\therefore C_v = \frac{Nk^2\pi^2 T}{2\epsilon_F(0)} \simeq \frac{\pi^2}{2} R \frac{T}{T_F} \quad -(12)$$

Specific heat C_v therefore varies linearly with temperature and vanishes at $T=0$ in conformity with third law of thermodynamics. In order to have an idea of the magnitude of the electronic contribution to the specific heat at room temperature, let us suppose that the Fermi temperature $T = 45000K$ (note that Fermi temperature are invariably of the order of $10^4 - 10^5 K$).

$$\therefore C_v = \frac{\pi^2}{2} \cdot R \cdot \frac{300}{45000} \simeq 0.03R$$

which is very small compared to the lattice contribution $3R$. The electronic specific heat, therefore, is not normally detected at room temperature.

In general, the low temperature specific heat of a metal is given by — $C_v = \gamma T + \beta T^3$

Now, pressure $p = \frac{2}{3}\frac{E}{V} = \frac{2}{3}\frac{N}{V}\left\{\frac{3}{5}\epsilon_F(0) + \frac{k^2T^2\pi^2}{4\epsilon_F(0)}\right\}$

$$= \frac{2}{5} n\epsilon_F(0)\left\{1+\frac{5}{12}\frac{k^2T^2\pi^2}{\epsilon_F(0)^2}\right\}$$

This shows that even at absolute zero, the pressure doesnot vanish. This is a consequence of Pauli exclusion principle which allows only one particle to have zero momentum. other particles having finite momentum give rise to zero point pressure.

A relativistic degenerate electron gas –

In a completely degenerate extreme relativistic electron gas, the energy of a particle is given by – $\epsilon = cp$ — (1).

$$\therefore \epsilon_F = cp_F = ch\left(\frac{3N}{4\pi(2s+1)V}\right)^{1/3} \quad — (2).$$

The total energy of the gas at absolute zero of temperature is –

$$E_0 = \frac{(2s+1)4\pi V}{h^3}\int_0^{p_F} cp^3\,dp.$$

$$= \frac{(2s+1)4\pi Vc}{h^3}\cdot\frac{p_F^4}{4}$$

$$= (2s+1)\pi Vch\left(\frac{3N}{4\pi(2s+1)V}\right)^{4/3}$$

$$= \frac{chN}{\pi^{1/3}(2s+1)^{1/3}}\left(\frac{3}{4}\right)^{4/3}\left(\frac{N}{V}\right)^{1/3}. \quad — (3)$$

$$P = -\frac{\partial E_0}{\partial V} = \frac{1}{3}\left(\frac{3}{4}\right)^{4/3}\cdot\frac{chN^{4/3}}{\pi^{1/3}(2s+1)^{1/3}V^{4/3}}$$

$$= \left(\frac{3}{\pi(2s+1)}\right)^{1/3} ch\left(\frac{N}{4V}\right)^{4/3}. \quad — (4)$$

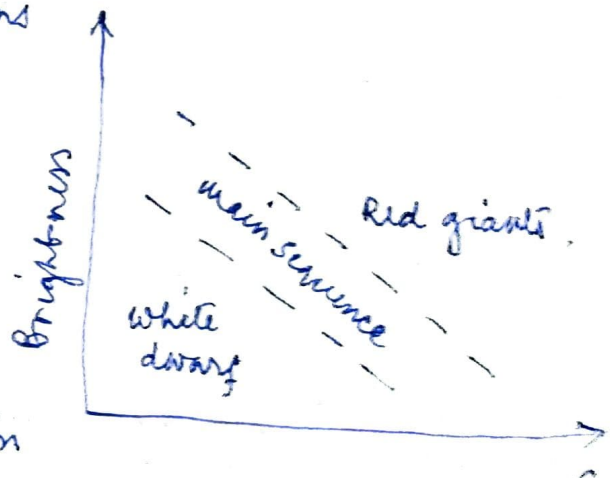
The pressure of an extreme relativistic gas, is, therefore, proportional to $n^{4/3}$.

Equation (3) and (6) leads to the relation, $PV = \frac{1}{3}E$. —(5)

It may be mentioned here that this relation is valid not only at absolute zero but all temperatures.

White dwarf and chandrasekhar limit

The white dwarf illustrates an application of the degenerate electron gas theory to a problem in stellar dynamics. white dwarf stars provide clues to the physical processes that take place during the evolutionary stages near the ends of stellar life times. If a plot is made of the brightness of stars against the predominant wavelength emitted, it will be found that most of the stars fall within a linear strip indicating that the brightness is proportional to x. Such a plot is known as the ~~Hertzsprung~~ Hertzsprung-Russel diagram. The sequence of stars is called the main sequence. There are, however, two glaring exceptions to this general rule: (i) the red giant stars which are huge and abnormally bright and (ii) the white dwarf stars which are — 'highly underluminous'. The companion of Sirius which is a white dwarf, has a mass about equal to that



of the sun but its luminosity is only 0.003 times that of sun. White dwarfs have stellar masses but much smaller diameters and hence are very dense. The gravitational red shift of light predicted by Einstein's general theory of relativity and measured in white dwarfs Sirius B confirmed the small size and high density of this stars. Existence of such compact stars could not be satisfactorily explained until the quantum statistical theory of the electron gas was worked out by Fermi and Dirac. It has been shown by Fowler and Chandrasekhar that such stellar configurations can be satisfactorily be described by treating the electrons within the stars as a completely degenerate relativistic Fermi gas. Chandrasekhar found that there is a critical stellar mass above which stable degenerate dwarfs cannot exist.

Idealized model of white dwarf -

From some typical data, an idealized model for a white dwarf may be constructed. Imagine a star with mass $M = 10^{33}$ grams and density 10^7 gm/cc and central temperature $T = 10^7$ K. Material content of white dwarfs is mostly helium and the temperature 10^7 K corresponds to a mean thermal energy of 10^3 eV, which is much larger in comparison with the energy required for ionizing a helium atom. Hence, the helium atoms in the star are expected to be in a state of complete ionization and the star may be regarded as a gas composed of N electrons and $N/2$ helium nuclei. Therefore, the mass of star is given by - $M = N(m + 2m_p) \approx 2Nm_p$. —(1)

where m is the mass of an electron and m_p is the mass of the proton. Mass of helium $\approx 4 m_p$. The electron density, therefore, is —

$$n = \frac{N}{V} = \frac{M/2m_p}{M/\rho} = \frac{\rho}{2m_p} \approx 10^{30} \text{ electrons/cc}.$$

Substituting this value for N/V in the expression for ϵ_F and using $\epsilon_F = kT_F$, the Fermi temperature comes to the order of 10^{10}K. Since this is much higher than the temperature of the star, the electron gas must be highly degenerate Fermi gas. The Fermi energy is of the order of 1 MeV which is comparable with the rest energy mc^2 of an electron, and, therefore, the dynamics of the electrons is relativistic. Though helium nuclei do not contribute significantly to the dynamics of the problem, and, hence, may be neglected, they do provide the gravitational attraction to hold the entire system together.

Let us now consider the equilibrium configuration our idealized model. Assuming the configuration to be spherical, the gas will exert a pressure $p_0(r)$ on the boundary of the sphere and any expansion will involve the expenditure of work. An adiabatic change in volume will cause a change dE_0 in energy given by

$$dE_0 = -p_0(r)\,dv = -p_0(r)\,4\pi r^2 dr \quad — (2).$$

If there were the only force present, the system would expand indefinitely. However, there is gravitational interaction, and, therefore there is a change in the gravitational self-energy as well. This change will be equal to $\frac{\alpha G M^2}{r^2} dr$, where M is the mass of the stellar matter

contained within a sphere of radius R, i.e., $M = 4\pi \int_0^R \rho r^2 dr$

G is the gravitational constant and α is a number whose value depends upon the spatial variation of $n = N/V$ inside the sphere.

If the configuration is in equilibrium, the change in its energy for an infinitesimal change in its size must be identically zero.

$$-P_0(r)\,4\pi r^2 + \frac{\alpha G M^2}{r^2} = 0, \quad \text{or,} \quad P_0(r) = \frac{\alpha G M^2}{4\pi r^4} \qquad \text{—— (3)}$$

It is evident that to find the relation between M and r we have to find the value of $P_0(r)$. The single particle energy of a relativistic particle is given by $E^2 = p^2c^2 + m^2c^4$ —— (4)

The ground state energy of the Fermi gas, therefore, is —

$$E_0 = 2\sum_p \sqrt{p^2c^2 + m^2c^4} = \frac{8\pi V}{h^3}\int_0^{p_F} p^2\sqrt{p^2c^2 + m^2c^4}\,dp \qquad \text{—— (5)},$$

where the factor 2 accounts for the spin degeneracy and p_F is the Fermi momentum $= h\left(\frac{3N}{8\pi V}\right)^{1/3}$.

Putting $x = p/mc$, $E_0 = \frac{8\pi V}{h^3} m^4c^5 \int_0^{x_F} x^2\sqrt{x^2+1}\,dx$ —— (6).

$$= \frac{8\pi V}{h^3} m^4c^5 f(x_F) \qquad \text{—— (7)}.$$

where $f(x_F) = \int_0^{x_F} x^2\sqrt{x^2+1}\,dx$. and $x_F = \frac{p_F}{mc} = \frac{h}{mc}\left(\frac{3}{8\pi}\cdot\frac{N}{V}\right)^{1/3}$. —— (8)

Since $\frac{N}{V} = \frac{M/2m_p}{\frac{4}{3}\pi R^3} = \frac{3M}{8\pi m_p R^3}$ —— (9)

$$x_F = \frac{h}{mc}\left(\frac{3}{8\pi}\cdot\frac{3M}{8\pi m_p R^3}\right)^{1/3} = \left(\frac{9h^3M}{64\pi^2 m_c^3 m_p R^3}\right)^{1/3} \qquad \text{—— (10)}.$$

The pressure exerted by the Fermi gas —

$$P_0 = -\frac{\partial E_0}{\partial V} = -\frac{8\pi m^4 c^5}{h^3} f(x_F) - \frac{8\pi m^4 c^5 V}{h^3} \frac{\partial f(x_F)}{\partial x_F} \cdot \frac{\partial x_F}{\partial V}$$

$$= -\frac{8\pi m^4 c^5}{h^3} f(x_F) + \frac{8\pi m^4 c^5}{3h^3} x_F^2 \sqrt{1+x_F^2} \left(\frac{3Nh^3}{8\pi m^3 c^3 V}\right)^{1/3} \quad \text{—(11)}$$

$$\simeq -\frac{8\pi m^4 c^5}{h^3} f(x_F) + \frac{8\pi m^4 c^5}{3h^3} x_F^2 \sqrt{1+x_F^2}$$

$$\simeq \frac{8\pi m^4 c^5}{h^3} \left[\frac{1}{3} x_F^3 \sqrt{1+x_F^2} - f(x_F)\right] \quad \text{—(12)},$$

(i) If $x_F \ll 1$, i.e., $\frac{p_F}{mc} \ll 1$, we would be dealing with a non-relativistic case.

$$\therefore \int_0^{x_F} x^2 \sqrt{1+x^2}\, dx = \int_0^{x_F} \left(x^2 + \frac{1}{2}x^4 + \dots\right) dx$$

$$\simeq \frac{x_F^3}{3} + \frac{x_F^5}{10} \quad \text{—(13)},$$

and $\frac{1}{3} x_F^3 \sqrt{1+x_F^2} \simeq \frac{x_F^3}{3} + \frac{x_F^5}{6}$ — (14),

where the higher terms are neglected. Substituting (13) and (14) in (12), $P_0 \simeq \frac{8\pi m^4 c^5}{15h^3} x_F^5 \simeq \frac{8\pi m^4 c^5}{15h^3} \left(\frac{9h^3 M}{64\pi^2 m^3 c^3 m_p R^3}\right)^{5/3}$

$\therefore$, From (3), $\frac{8\pi m^4 c^5}{15h^3} \left(\frac{9h^3 M}{64\pi^2 m^3 c^3 m_p R^3}\right)^{5/3} \simeq \frac{\alpha G M^2}{4\pi R^4}$ — (15), — (16)

Hence $M^{1/3} R = \frac{32\pi^2 h^2}{15 M \alpha G} \left(\frac{9}{64\pi^2 m_p}\right)^{5/3} = \text{constant}$ — (17).

(ii) If on the other hand $x_F \gg 1$, the case is extreme relativistic.

In this case $f(x_F) = \int_0^{x_F} x^2\sqrt{1+x^2}\,dx = \int_0^{x_F} x^2\left(x + \frac{1}{2x} + \dots\right)dx$

$$\approx \frac{x_F^4}{4} + \frac{x_F^2}{4} \quad \text{— (18)}.$$

and $\frac{1}{3}x_F^3\sqrt{x_F^2+1} = \frac{x_F^4}{3} + \frac{x_F^2}{6}$ — (19).

$$\therefore P_0 \approx \frac{8\pi m^4c^5}{h^3}\left[\frac{x_F^4}{3} + \frac{x_F^2}{6} - \frac{x_P^4}{4} - \frac{x_P^2}{4}\right] \approx \frac{8\pi m^4c^5}{12h^3}(x_F^4 - x_P^4)$$

— (20).

$$\therefore \frac{8\pi m^4c^5}{12h^3}(x_F^4 - x_F^2) = \frac{\alpha}{4}\cdot\frac{GM^2}{\pi R^4} \quad \text{— (21)}.$$

or, $$\frac{8\pi m^4c^5}{12h^3}x_F^2(x_F^2-1) = \frac{\alpha GM^2}{4\pi R^4} \quad \text{— (22)}.$$

$$\therefore \frac{8\pi m^4c^5}{12h^3}\left(\frac{9h^3}{64\pi^2m^3c^3m_p}\right)^{2/3}\frac{M^{2/3}}{R^2}\left\{\left(\frac{9h^3}{64\pi^2m^3c^3m_p}\right)^{2/3}\frac{M^{2/3}}{R^2} - 1\right\}$$

$$= \frac{\alpha GM^2}{4\pi R^4} \quad \text{— (23)}.$$

Putting $x = \frac{8\pi m^4c^5}{12h^3}$ and $Y = \left(\frac{9h^3}{64\pi^2m^3c^3m_p}\right)^{2/3}$.

eqn. (23) can be written as — $xy\left(Y\frac{M^{2/3}}{R^2} - 1\right) = \frac{\alpha M^{4/3}G}{4\pi R^2}$ — (24)

$$\therefore \frac{1}{R^2}\left(YM^{2/3} - \frac{\alpha G}{4\pi xy}M^{4/3}\right) = 1 \quad \text{— (25)},$$

and, hence, $R = M^{1/3}\left(Y - \frac{\alpha G}{4\pi xy}M^{2/3}\right)^{1/2}$ — (26).

R must be positive or zero.

$$\therefore Y \geq \frac{\alpha G}{4\pi xy}M^{2/3}$$

i.e., $M^{2/3} < \frac{4\pi x y^2}{\alpha G}$ ——— (27).

or, $M < \left(\frac{4\pi x y^2}{\alpha G}\right)^{3/2}$ ——— (28)

$$M_{Ch}\ \alpha = \left(\frac{4\pi x y^2}{\alpha G}\right)^{3/2} = \left(\frac{8\pi^2 m^4 c^5}{3h^3 \alpha G}\right)^{3/2} \left(\frac{9h^3}{64\pi^2 m^3 c^3 m_p}\right)^2$$ ——— (29),

is the limiting mass, M_{Ch} is known as the <u>chandrasekhar limit</u>. Detailed investigation of chandrasekhar lead to the result — $M_{Ch} = 1.44 M_\odot$, where $M_\odot$ is the mass of the sun. Stars that end their evolution with masses in excess of this limit have gravitational self-attraction too great to be counterbalanced by the pressure of the degenerate electrons and are doomed to ultimate collapse.