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Electric field lines - flux -

Field lines begin on positive charges and end on negative ones; they cannot terminate midair, but can extend till infinity. However, field lines can never cross since, at the intersection, the field lines will have two directions at once.

The electric flux of $\vec{E}$, which is a measure of the number of field lines passing through the surface, is given by -

$$\boxed{\Phi_E = \int_S \vec{E}\cdot d\vec{a}} \quad \text{——} (1)$$

The flux through a 'closed' surface is a measure of the total charge inside. For the field lines that originate on a positive charge must either pass out through the surface or else terminate on a negative charge inside. On the other hand, a charge outside the surface will contribute nothing to the total flux, since its field lines pass in one side and out the other. In case of a point charge q, at the origin, the flux of $\vec{E}$ through a sphere of radius r is -

$$\oint \vec{E}\cdot d\vec{a} = \int \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r^2}\hat{r}\right)\cdot\left(r^2 \sin\theta\, d\theta\, d\phi\, \hat{r}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\cdot q \cdot \int_0^{\pi} \sin\theta\, d\theta \cdot \int_0^{2\pi} d\phi\ (\hat{r}\cdot\hat{r})$$

$$= \frac{1}{4\cancel{\pi}\varepsilon_0}\cdot q \cdot \cancel{2}\cdot 2\cancel{\pi}\cdot 1 = \frac{q}{\varepsilon_0}. \quad \text{——} (2).$$

For several charges q_i, $\vec{E} = \sum_i \vec{E_i}$

The flux through a surface that encloses them all, then, is -

$$\oint \vec{E}\cdot d\vec{a} = \sum_i \oint \vec{E_i}\cdot d\vec{a} = \sum_i \frac{q_i}{\varepsilon_0}$$

For any closed surface, then, $\boxed{\oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{enclosed}}{\varepsilon_0}}$ — (3)

This is the quantitative statement of Gauss's law. This is an integral equation, but can be readily converted to differential form, by applying the divergence theorem,

$$\oint \vec{E}\cdot d\vec{a} = \oint_V (\nabla\cdot\vec{E})\,dv$$

Rewriting Q_{enc} in terms of the charge density ρ, we have—

$$Q_{enc} = \int_V \rho\, dv.$$

So, Gauss's law becomes —

$$\oint (\nabla\cdot\vec{E})\,dv = \frac{\int_V \rho\, dv}{\varepsilon_0}$$

Since this holds for any volume, $\boxed{\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}}$ — (4)

This is the Gauss's law in differential form.

Problem 2.9 Suppose the electric field in some region is found to be $\vec{E} = kr^3\hat{r}$, in spherical coordinates (k is some constant)

(a) find the charge density ρ

(b) find the total charge contained in a sphere of radius R, centred at the origin.

Ans:- (a) $\nabla\cdot\vec{E} = \rho/\varepsilon_0$ so, $\rho = \varepsilon_0(\nabla\cdot\vec{E})$

Here $\vec{E} = kr^3\hat{r}$, so only the radial component exist (in spherical coords.).

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~~Taking only the radial component,~~ $\cancel{\rho = \epsilon_0 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial k r^3}{\partial r}\right)}$

$$\nabla \cdot \vec{f} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 f_r) + \frac{1}{r}\sin\varphi \frac{\partial f_\theta}{\partial \theta} + \frac{1}{r\sin\varphi}\frac{\partial}{\partial \varphi}(\sin\varphi f_\varphi)$$

taking only the radial component, $\rho = \epsilon_0 \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 . kr^3)\right]$

or, $\rho = \epsilon_0 . \frac{1}{r^2} . k . \frac{\partial r^5}{\partial r}$

$$= \epsilon_0 k . \frac{1}{r^2} . 5 r^4 r^2 = 5 \epsilon_0 k r^2.$$

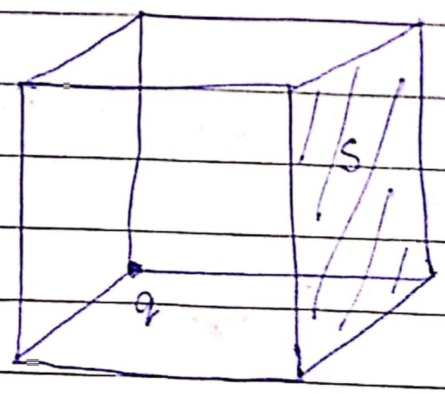
(b) $$Q_{enc} = \int \rho \, dv = \int_0^R 5\epsilon_0 k r^2 . 4\pi r^2 dr$$

$$= 20 \epsilon_0 k \pi \int_0^R r^4 dr = 20 . \epsilon_0 \pi . k . \frac{R^5}{5} = 4\epsilon_0 \pi k R^5.$$

also, $\frac{Q_{enc}}{\epsilon_0} = \oint \vec{E} . d\vec{a}$

or, $$Q_{enc} = \epsilon_0 . \int \vec{E} . d\vec{a} = \epsilon_0 (kR^3) . (4\pi R^2)$$
$$= 4\pi \epsilon_0 k R^5$$

Problem 2.10 A charge q sits at the back corner of a cube, as shown in figure alongside. What is the flux $\vec{E}$ through the shaded side?



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ans:- From the symmetry consideration, if the charge was at the centre of the cube, each face would receive equal amount of flux. But since the charge is at one side corner, to make it at the centre, we can consider a total of seven other cubes surrounding it, to make it at the centre. Then each of the cubes faces would experience the same flux. There are six faces to the 'super-cube' consisting of 8 cubes. Each side has area of $(2a)^2 = 4a^2$.

So, $E.(6 \times 4a^2) = q/\varepsilon_0$.

or, $E.\hat{a} = \frac{1}{24} \cdot \frac{q}{\varepsilon_0} \cdot \frac{q}{\cancel{q}} = \frac{1}{24} \cdot \left(\frac{q/\varepsilon_0}{\cancel{q}}\right)$

which is basically the flux through a single cube surface,

So, $E = \frac{1}{24}\left(\frac{q}{\varepsilon_0}\right)$

example 2.2 Find the field outside a uniformly charged solid sphere of radius R and total charge q.

ans:- Draw a spherical surface at radius $r > R$, this is the Gaussian surface here. $\oint_S \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$

or, $E \cdot 4\pi r^2 = q/\varepsilon_0$

or, $\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$.

i.e., the field outside the sphere is exactly the same as it would be if all the charges had been concentrated at the centre of the sphere.

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Gauss's law works best when there is some sort of symmetry. There are three kinds of symmetry that works –

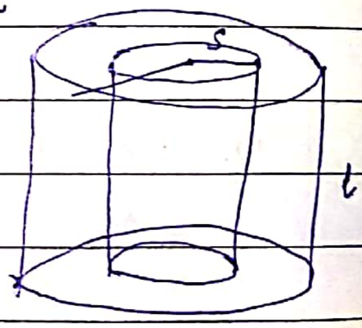
1. Spherical symmetry – Make the Gaussian surface a concentric sphere.

2. cylindrical symmetry – Make the Gaussian surface a coaxial cylinder

3. Plane symmetry – use a Gaussian 'pill box', which straddles the surface.

Example 2.3 A long cylinder carries a charge density that is proportional to the distance from the axis: $\rho = ks$, for some constant k. Find the electric field inside this cylinder.

ans:- Draw a Gaussian cylinder of length l and radius s. For this surface,



$$\oint_S \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} q_{enc}$$

The enclosed charge is,

$$q_{enc} = \int \rho \, d\tau = \int (ks')(s' ds' d\phi \, dz)$$

$$= k \int_0^s s'^2 ds' \int_0^{2\pi} d\phi . \int_0^l dz$$

$$= k . \left[\frac{s'^3}{3}\right]_0^s . 2\pi . l = k . 2\pi l . \frac{s^3}{3} .$$

Now, due to the symmetry, $\vec{E}$ must point radially outward, so for the curved portion of the Gaussian cylinder, we have –

$$\int \vec{E} \cdot d\vec{a} = \int |E| da = E . 2\pi s l .$$

The two ends donot contribute to the net flux. Hence,

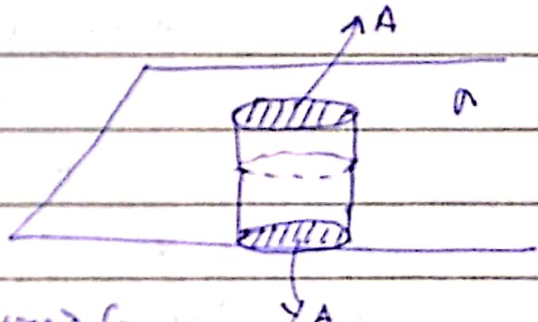
$$E \cdot 2\pi s l = \frac{2}{3} \cdot \frac{1}{\varepsilon_0} \cdot \pi k l s^3$$

or, $$\vec{E} = \frac{1}{3\varepsilon_0} \cdot k s^2 \hat{s}.$$

example 2.4 An infinite plane carries a uniform charge density σ. Find its electric field.

ans:- Draw a Gaussian pillbox extending equal distance above and below the plane.

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}.$$



in this case, $Q_{enc} = \sigma A$, where A is the area of the lid of the pillbox. By symmetry, $\vec{E}$ points away from the plane (upward for point above, downward for points below). Thus, the top and bottom surface yeilds,

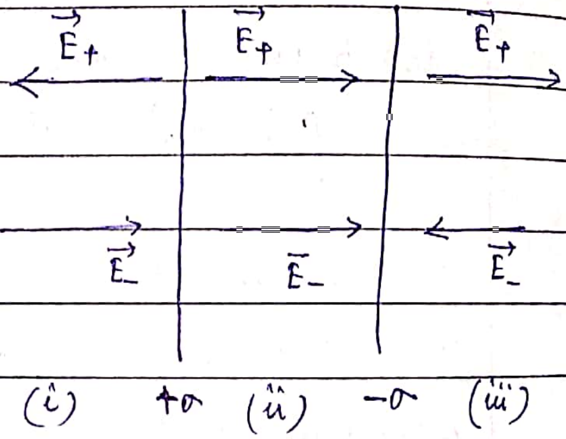
$$\int \vec{E} \cdot d\vec{a} = 2AE.$$

The sides contribute nothing. Thus, $2AE = \frac{\sigma A}{\varepsilon_0}$

or, $\vec{E} = \frac{\sigma}{2\varepsilon_0} \hat{n}$. $\hat{n}$ = unit vector pointing away from the surface.

It may seem surprising at first that the field of an infinite plane is independent of how far away we are. To justify this fact, we see that, as we move further and farther away from the plane, more and more charges comes into our field of view, and this compensates for the diminishing influence of any particular piece.

example 2.5 Two infinite parallel planes carry equal but opposite uniform charge densities $\pm\sigma$. Find the field in each of the three regions: (i) to the left of both, (ii) between them and (iii) to the right of both.



ans:- The left plate produces a field $\sigma/2\varepsilon_0$ which points away from it - to the left in region (i) and to the right in region (ii) and (iii). The right plate being negatively charged, produces a field $\sigma/2\varepsilon_0$ which points towards it in region (i) & (ii) - towards the right, to the left in region (iii).

$\therefore$ Field in (i) $= \frac{\sigma}{2\varepsilon_0} - \frac{\sigma}{2\varepsilon_0} = 0$

(ii) $= \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$.

(iii) $= \frac{\sigma}{2\varepsilon_0} - \frac{\sigma}{2\varepsilon_0} = 0$.

Problem 2.11 use Gauss's law to find the electric field inside and outside a spherical shell of radius R, which carries a uniform charge density σ.

ans:- Inside, $\oint \vec{E} \cdot d\vec{a} = E(4\pi r^2) = \frac{Q_{enc}}{\varepsilon_0} = 0$

$$\text{or,} \quad \vec{E} = 0.$$

. outside, $\oint \vec{E} \cdot d\vec{a} = E(4\pi r^2) = \frac{Q_{enc}}{\varepsilon_0} = \frac{\int \sigma ds}{\varepsilon_0} = \frac{4\pi R^2 \sigma}{\varepsilon_0}$,

$$\text{or,} \quad E = \frac{4\pi R^2 \sigma}{4\pi r^2 \varepsilon_0} = \frac{R^2 \sigma}{r^2 \varepsilon_0}.$$

$$\vec{E} = \frac{R^2 \sigma}{\varepsilon_0} \cdot \frac{1}{r^2} \hat{r}.$$

Problem 2.12 use Gauss's law to find the electric field inside a uniformly charged sphere with volume charge density ρ.

ans:- $\oint \vec{E} \cdot da = \frac{q_{enc}}{\varepsilon_0} = \frac{\frac{4}{3}\pi r^3 \rho}{\varepsilon_0}$ (for $r < R$, $R =$ radius of sphere)

$$E \cdot 4\pi r^2 = \frac{4/3\, \pi r^3 \rho}{\varepsilon_0}$$

$$\text{or,} \quad \vec{E} = \frac{r\rho}{3\varepsilon_0} \hat{r},$$

in terms of the total charge Q, $Q = \frac{4}{3}\pi R^3 \rho$, and the fraction of charge contained within r ($r < R$) is $= \frac{\frac{4}{3}\pi r^3 \rho}{\frac{4}{3}\pi R^3 \rho} Q = \frac{r^3}{R^3} Q$

and, $E \cdot 4\pi \cancel{r^2} = \frac{\cancel{r^3}\, r}{R^3} Q \Rightarrow \vec{E} = \frac{rQ}{4\pi R^3} \hat{r}$,

Problem 2.13 Find the electric field a distance s from an infinitely long straight wire, which carries an uniform line charge λ.

ans:- $\oint E \cdot da = q/\varepsilon_0$.

$$E \cdot 2\pi s \cdot \cancel{l} = \frac{\lambda \cancel{l}}{\varepsilon_0}.$$

or, $$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 \cdot s}\hat{s}.$$

Problem 2.14 Find the electric field inside a sphere which carries a charge density proportional to the distance from the origin $\rho = kr$, for some constant k.

ans:- $$\oint \vec{E}\cdot da = \frac{q_{enc}}{\varepsilon_0} = \frac{\int \rho\, dv}{\varepsilon_0} = \frac{\int \rho \cdot (r^2 \sin\theta\, d\theta\, d\phi\, dr)}{\varepsilon_0}$$

or, $$E \cdot 4\pi r^2 = \frac{k}{\varepsilon_0}\int_0^r r^3 dr \cdot \int_0^\pi \sin\theta\, d\theta \cdot \int_0^{2\pi} d\phi = \frac{k \cdot 2 \cdot 2\pi}{\varepsilon_0}\int_0^r r^3 dr$$

or, $$E \cdot 4\cancel{\pi}\cancel{r^2} = \frac{4\cancel{\pi}k}{\varepsilon_0} \cdot \frac{r^4}{\cancel{4}}\ r^2 \qquad (r<R)$$

or, $$\vec{E} = \frac{kr^2}{4\varepsilon_0}\hat{r}, = \frac{1}{4\pi\varepsilon_0} \cdot \pi k r^2 \hat{r}.$$

or, $$\vec{E} = \frac{1}{4\pi\varepsilon_0} \cdot (\pi k r^2)\hat{r}.$$

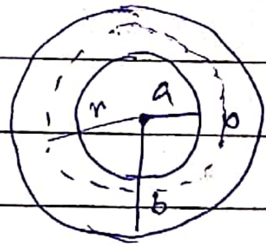
Problem 2.15 A hollow spherical shell carries charge density

$$\rho = k/r^2$$

in the region $a \le r \le b$. Find the electric field in the three regions — (i) $r<a$, (ii) $a<r<b$, (iii) $r>b$. Plot E vs. r.

ans:- (i) Since no charge is enclosed in $r<a$,

$\oint \vec{E}\cdot d\vec{a} = \frac{q}{\varepsilon_0} = 0$, because $q=0$, so, $E=0$



(ii) $\oint \vec{E}\cdot d\vec{a} = q/\varepsilon_0$

$$E\cdot 4\pi r^2 = \frac{1}{\varepsilon_0}\int_a^r \frac{k}{r^2}\cdot(r^2\sin\theta\, d\theta\, d\varphi\, dr) = \frac{k}{\varepsilon_0}\cdot\int_0^\pi \sin\theta\, d\theta\int_0^{2\pi} d\varphi \int_a^r dr.$$

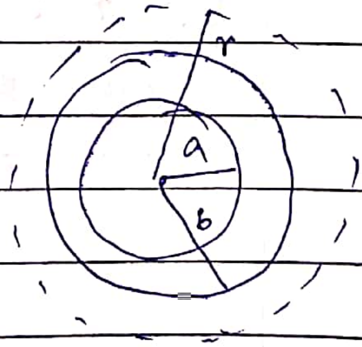
or, $E\cdot 4\pi r^2 = \frac{k}{\varepsilon_0}\cdot 2\cdot 2\pi\cdot(r-a)$

or, $\vec{E} = \frac{k}{4\pi\varepsilon_0}\cdot\frac{4\pi(r-a)}{r^2}\hat{r}.$

(iii) $q_{enc} = \int_a^b \frac{k}{r^2}\cdot(r^2\sin\theta\, d\theta\, d\varphi\, dr)$

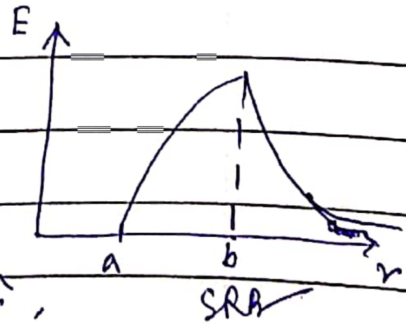
$= k\cdot 2\cdot 2\pi\cdot(b-a).$

$= 4\pi k(b-a)$



$\therefore E\cdot 4\pi r^2 = \frac{4\pi k(b-a)}{\varepsilon_0}$

or, $E = \frac{1}{4\pi\varepsilon_0}\frac{4\pi k(b-a)}{r^2}$



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or, $\vec{E} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{4\pi k(b-a)}{r^2}\hat{r}.$

Problem 2.16 A long coaxial cable carries an uniform volume charge density ρ on the inner cylinder (radius a), and an uniform surface charge density on the outer cylindrical shell (radius b). This surface charge is negative and just the right magnitude so that the cable as a whole is electrically neutral. Find the electric field in each of the three regions — (i) $r<a$, (ii) $a<r<b$ and (iii) $r>b$.

ans:- (i) $r<a$, $\quad E \cdot 2\cancel{\pi} \cancel{r} \cancel{l} = \dfrac{\rho \cdot \cancel{\pi} r^{\cancel{2}} \cancel{l}}{\epsilon_0}$

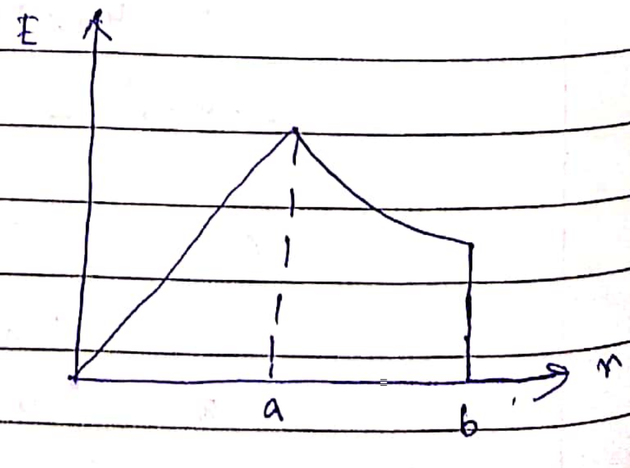
$$\Rightarrow \vec{E} = \frac{\rho r}{2\epsilon_0}\hat{r}.$$

(ii) $a<r<b$, $\quad E \cdot 2\cancel{\pi} r \cancel{l} = \dfrac{\rho \cdot \cancel{\pi} a^2 \cancel{l}}{\epsilon_0}$.

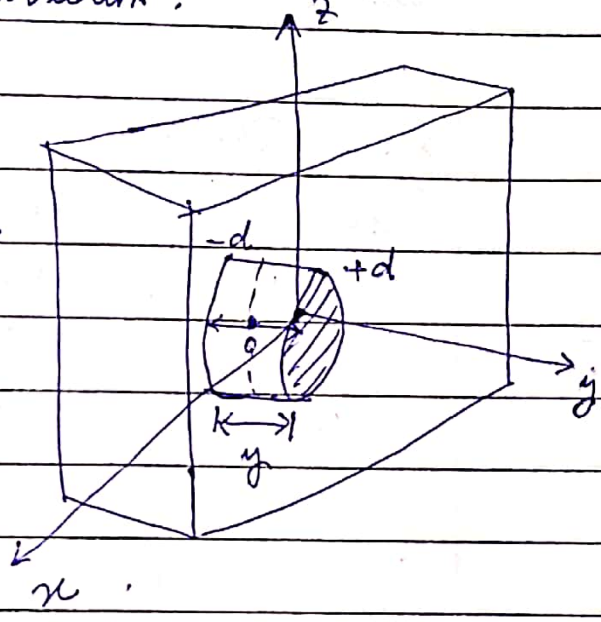
$$\text{or, } \vec{E} = \frac{\rho a^2}{2\epsilon_0 r}\hat{r}.$$

(iii) $r>b$, since the volume charge inside a cancels out the surface charge, hence, for any point outside b, the net charge enclosed is zero, $q_{enc} = 0$.

So, $E = 0$.



Problem 2.17 An infinite plane slab of thickness 2d, carries an uniform volume charge density ρ. Find the electric field as a function of y, where $y=0$ at the centre. Plot E vs. y, calling E positive when it points in the $+y$ direction and negative when it points in $-y$ direction.



Ans:- By symmetry, the electric field will be in the y-direction. So, in the x-z plane $E=0$.

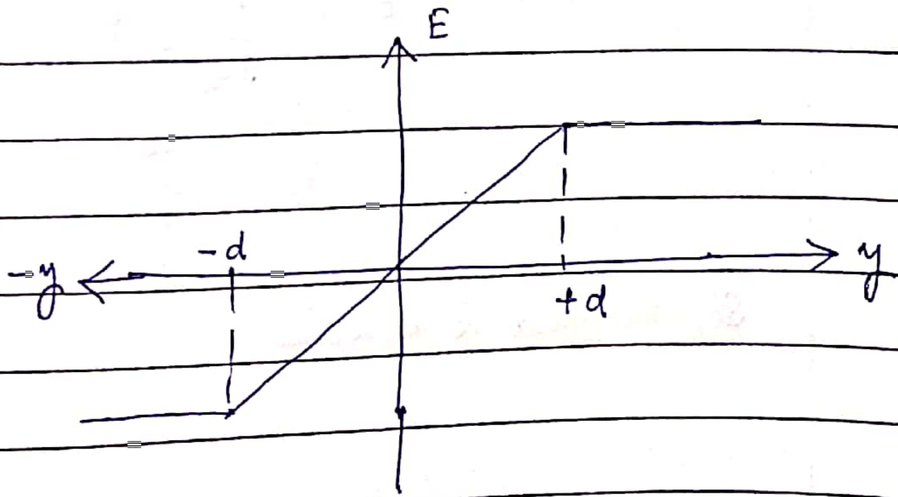
$$\oint \vec{E}\cdot da = \frac{q_{enc}}{\varepsilon_0}, \text{ for } |y|<d$$

$$\text{or, } E \cdot A = \frac{\rho . A y}{\varepsilon_0}$$

$$\text{or, } \vec{E} = \frac{\rho y}{\varepsilon_0}\hat{y} \rightarrow \text{for distances } -d<y<d$$

For $|y|>d$, $\quad E \cdot A = \frac{\rho . A . 2d}{\varepsilon_0}$

$$\text{or, } \vec{E} = \frac{\rho d}{\varepsilon_0}.\hat{y}. \rightarrow \text{for } \begin{cases} y<-d \\ y>d \end{cases}$$



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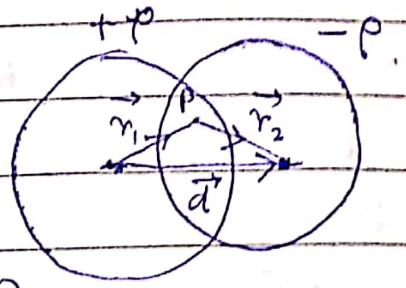
Problem 2.18 Two spheres, each of radius R and carrying uniform charge densities $+\rho$ and $-\rho$ respectively, are placed so that they partially overlap. Call the vector from the positive centre to the negative centre $\vec{d}$, Show that the field in the region of overlap, is constant, and find its value.



ans:- The electric field inside a sphere is given by,

$$E \cdot 4\pi r^2 = \frac{1}{\varepsilon_0}\frac{4}{3}\pi r^3 \rho \quad (r<R)$$

or, $$\vec{E} = \frac{\rho r}{3\varepsilon_0}\hat{r}.$$

So, the field inside the positive charge sphere is, $\vec{E}_+ = \frac{\rho}{3\varepsilon_0}\vec{r}_+$ where $\vec{r}_+$ is the vector from the +ve charged sphere's centre to some point P, where they overlap. Similarly, the field of the negative charge sphere is $\vec{E}_- = -\frac{\rho}{3\varepsilon_0}\vec{r}_-$. where $\vec{r}_-$ = radius vector from the centre of -ve charged sphere to P. Then, $\vec{d} = \vec{r}_+ - \vec{r}_-$

and, $$\vec{E} = \frac{\rho}{3\varepsilon_0}(\vec{r}_+ - \vec{r}_-)$$

$$= \frac{\rho \vec{d}}{3\varepsilon_0}.$$

So, the field is constant.

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calculating the Curl of $\vec{E}$

For a point charge q at the origin, $\vec{E} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r^2}\hat{r}$
in spherical coordinates, $d\vec{l} = dr\hat{r} + r d\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$.

$$\therefore \vec{E}\cdot d\vec{l} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r^2}[\hat{r} \cdot (dr\hat{r} + r d\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi})]$$

$$= \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r^2} dr.$$

$$\int_a^b \vec{E}\cdot d\vec{l} = \frac{1}{4\pi\epsilon_0}\int_a^b \frac{q}{r^2} dr = \frac{q}{4\pi\epsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

where r_a is the distance from the origin to the point $\vec{a}$ and r_b is the distance to $\vec{b}$. The integral around a closed path is evidently zero (for then, $r_a = r_b$); $\oint \vec{E}\cdot d\vec{l} = 0$.

Now applying stokes theorem, $\oint_S (\nabla\times\vec{E})\cdot\hat{n} ds = \oint_l \vec{E}\cdot d\vec{l} = 0$

or, $\boxed{\nabla\times\vec{E} = 0}$

If we have many charges, the principle of superposition states that the total field is a vector sum of their individual fields:

$$\vec{E} = \vec{E_1} + \vec{E_2} + \cdots$$

So, $\nabla\times\vec{E} = (\nabla\times\vec{E_1}) + (\nabla\times\vec{E_2}) + \cdots = 0.$

Electric potential –

Because $\nabla\times\vec{E} = 0$, the line integral of $\vec{E}$ around any closed loop is zero. Because $\oint \vec{E}\cdot d\vec{l} = 0$, the line integral of $\vec{E}$ from $\vec{a}$ to point $\vec{b}$ is the same for all paths. Because the line integral is independent of path, we can define a function, $V(\vec{r}) \equiv \int_O^{r} \vec{E}\cdot d\vec{l}$.
where O is some standard reference point. V then depends only on point $\vec{r}$. It is called electric potential.

Evidently, the potential difference between two points $\vec{a}$ and

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$\vec{b}$ is: $\quad V(\vec{b}) - V(\vec{a}) = -\int_{0}^{\vec{b}} \vec{E}\cdot d\vec{l} + \int_{0}^{\vec{a}} \vec{E}\cdot d\vec{l}$

$$= -\int_0^b \vec{E}\cdot d\vec{l} - \int_a^0 \vec{E}\cdot d\vec{l} = -\int_a^b \vec{E}\cdot d\vec{l}.$$

Now that the fundamental theorem of ~~gradi~~ gradient states that –

$$V(\vec{b}) - V(\vec{a}) = \int_a^b (\nabla V)\cdot d\vec{l}$$

So,

$$\int_a^b (\nabla V)\cdot d\vec{l} = -\int_a^b \vec{E}\cdot d\vec{l}$$

Since, this is true for any point $\vec{a}$ and $\vec{b}$, the integrands must be equal –

$$\boxed{\vec{E} = -\nabla V}$$

Problem 2.20 One of these is an impossible electrostatic field, which one?

(a) $\vec{E} = k\,[xy\,\hat{x} + 2yz\,\hat{y} + 3xz\,\hat{z}]$

(b) $\vec{E} = k\,[y^2\,\hat{x} + (2xy + z^2)\,\hat{y} + 2yz\,\hat{z}]$

Here k is a constant with the appropiate units.

ans:- (a) $\nabla\times\vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3xz \end{vmatrix} \neq 0$. So, this cannot be the electric field.

(b) $\nabla\times\vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy+z^2 & 2yz \end{vmatrix} = 0$. So this is a possible electric field.

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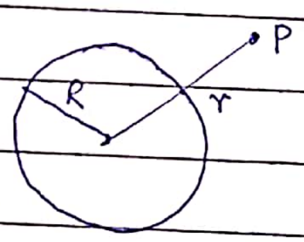
Talking about the origin O, there is a natural spot to use for O in electrostatics — and that is a point infinitely far from the charge. Ordinarily, we set the zero of potential at infinity. Since $V(O) = 0$, choosing a reference point is equivalent to selecting a place where V is to be zero.

Example 2.6 Find the potential inside and outside a spherical shell of radius R, which carries a uniform surface charge.

ans:- From Gauss's law, the field outside is,

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2}\,\hat{r}$$



where q is the total charge on the sphere.
The field inside is zero. For points outside the sphere $(r > R)$

$$V(r) = -\int_{\infty}^{r} \vec{E}\cdot d\vec{l} = -\frac{1}{4\pi\varepsilon_0}\int_{\infty}^{r} \frac{q}{r'^2}\,dr' = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r'}\Big|_{\infty}^{r}$$

$$= \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r}$$

To find the potential inside the sphere $(r < R)$, we must break the integral into two sections, using in each region the field that prevails there:

$$V(r) = -\frac{1}{4\pi\varepsilon_0}\int_{\infty}^{R} \frac{q}{r'^2}\,dr' - \int_{R}^{r} (0)\,dr' = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r'}\Big|_{\infty}^{R} + 0$$

or, $V(r) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{R}$

Notice that the potential is not zero inside the shell, even though the

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field is. V is a constant in this region, so that $\nabla V = 0$.

Problem 2.21 Find the potential inside and outside a uniformly charged solid sphere whose radius is R and whose total charge is q. Compute the gradient of V in each region, and check that it yields the correct field. Sketch V(r).

ans:- $V(r) = -\int_{\infty}^{r} \vec{E}\cdot d\vec{l}$

Now, outside the sphere $r > R$, $\vec{E} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r^2}\hat{r}$, and,

inside the sphere $r < R$, $\vec{E} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{qr}{R^3}\hat{r}$

So, for $r > R$, $V(r) = -\int_{\infty}^{r}\left(\frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r^2}\right)dr = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r}\Big|_{\infty}^{r}$

$$= \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{r}$$

For $r < R$, $V(r) = -\int_{\infty}^{R}\left(\frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r^2}\right)dr - \int_{R}^{r}\left(\frac{1}{4\pi\varepsilon_0}\cdot\frac{qr}{R^3}\right)dr$

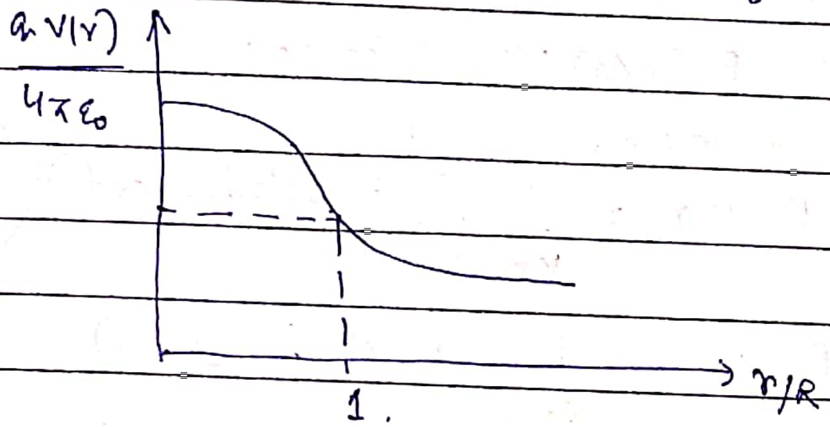
$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{R} - \frac{1}{R^3}\left(\frac{r^2-R^2}{2}\right)\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{2R}\left(3 - \frac{r^2}{R^2}\right).$$

when $r < R$, $\nabla V = \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{2R}\cdot\frac{\partial}{\partial r}\left(3 - \frac{r^2}{R^2}\right)\hat{r} = \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{2R}\cdot\left(\frac{-2r}{R^2}\right)\hat{r}$

$$= -\frac{q}{4\pi\varepsilon_0} \cdot \frac{r}{R^3}\,\hat{r}$$

when, $r > R$, $\nabla V = \frac{q}{4\pi\varepsilon_0} \cdot \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{r} = -\frac{q}{4\pi\varepsilon_0} \cdot \frac{1}{r^2}\,\hat{r}$



Problem 2.22 Find the potential a distance r from an infinitely long straight wire that carries a uniform line charge λ. Compute the gradient of your potential, and check that it yields the correct field.

ans:- For a long wire carrying charge density λ, $\vec{E} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{2\lambda}{r}\,\hat{r}$

in this case we cannot set the reference point at ∞, since the charge itself extends to ∞. Let us set this at say $r = a$.

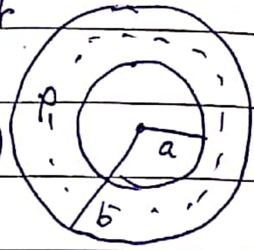
$$V(r) = -\int_a^r \frac{1}{4\pi\varepsilon_0} \cdot \frac{2\lambda}{r}\,dr = -\frac{1}{4\pi\varepsilon_0} \cdot 2\lambda \int_a^r \frac{dr}{r}$$

$$= -\frac{1}{4\pi\varepsilon_0} \cdot 2\lambda \ln\left(\frac{r}{a}\right) = \frac{1}{4\pi\varepsilon_0} \cdot 2\lambda \ln\left(\frac{a}{r}\right).$$

$$\nabla V = -\frac{1}{4\pi\varepsilon_0} \cdot 2\lambda \frac{\partial}{\partial r}\left(\ln\left(\frac{a}{r}\right)\right)\hat{r} = -\frac{1}{4\pi\varepsilon_0} \cdot \frac{2\lambda}{r}\,\hat{r}$$

Problem 2.23 For a hollow spherical shell carrying charge density $\rho = k/r^2$, k = constant in the region $a \le r \le b$, find the potential at the centre, using infinity as your reference point.

ans:- $E\,(r<a) = 0$, since no charge exist



for $a < r < b$,

$$E \cdot 4\pi r^2 = \frac{1}{\varepsilon_0} \int_0^{\pi}\int_0^{2\pi}\int_a^{r} \frac{k}{r^2}\left(r^2 \sin\theta \, d\theta \, d\phi \, dr\right)$$

or, $E \cdot \cancel{4\pi} r^2 = \frac{1}{\varepsilon_0} \cdot k \cdot \cancel{2} \cdot \cancel{2\pi} \cdot (r-a)$

or, $\vec{E} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{4\pi k(r-a)}{r^2}\, \hat{r}$

and, for $r > b$, $E \cdot 4\pi r^2 = \frac{1}{\varepsilon_0} \cdot \int_0^{\pi}\int_0^{2\pi}\int_a^{b} \frac{k}{r^2}\left(r^2 \sin\theta \, d\theta \, d\phi \, dr\right)$

or, $E \cdot \cancel{4\pi} r^2 = \frac{1}{\varepsilon_0}\, k\, \cancel{2} \cdot 2\cancel{\pi} \cdot (b-a)$

or, $\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{4\pi k(b-a)}{r^2}\, \hat{r}$

ie, $E = 0$ for $r < a$

$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{4\pi k(r-a)}{r^2}\, \hat{r}, \quad a < r < b$$

$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{4\pi k(b-a)}{r^2}\, \hat{r}, \quad r > b$$

To find the electric potential, at $r = 0$,

$$V = -\int_{\infty}^{0} \vec{E} \cdot \vec{dl} = -\int_{\infty}^{b} \frac{1}{4\pi\varepsilon_0} \cdot \frac{4\pi k(b-a)}{r^2}\, dr \; -$$

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$$\int_b^a \frac{1}{4\pi\varepsilon_0} \cdot \frac{4\pi k(\cancel{b}-\cancel{a})}{r^2}\,dr - \int_a^0 0 \cdot dr .$$

$$a, \quad V\Big|_{r=0} = \frac{1}{4\pi\varepsilon_0} \cdot 4\pi k(b-a)\left[\frac{1}{r}\right]_\infty^b - \int_b^a \cdot \frac{4\pi k}{4\pi\varepsilon_0}\left[\frac{1}{r} - \frac{a}{r^2}\right]dr$$

$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{4\pi k(b-a)}{b} - \frac{4\pi k}{4\pi\varepsilon_0} \ln r\Big|_b^a - \frac{4\pi k a}{4\pi\varepsilon_0} \cdot \left[\frac{1}{r}\right]_b^a$$

$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{4\pi k(b-a)}{b} - \frac{4\pi k \ln\left(\frac{a}{b}\right)}{4\pi\varepsilon_0} - \frac{4\pi k a}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$\Rightarrow \frac{4\pi k}{4\pi\varepsilon_0}\left[1 - \frac{a}{b} - \ln\left(\frac{a}{b}\right) - \frac{a}{a} + \frac{a}{b}\right]$$

$$= \frac{4\pi k}{4\pi\varepsilon_0}\left[\cancel{1} - \cancel{\frac{a}{b}} - \ln\left(\frac{a}{b}\right) - \cancel{1} + \cancel{\frac{a}{b}}\right]$$

$$= -\frac{4\pi k}{4\pi\varepsilon_0} \ln\left(\frac{a}{b}\right) = \frac{k}{\varepsilon_0} \ln\left(\frac{b}{a}\right)$$

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Problem 2.24 A long coaxial cable carries a uniform ~~surface~~ volume charge density ρ on the inner cylinder (radius a), and a uniform surface charge density on the outer cylindrical shell (radius b). This surface charge is negative and of just the right magnitude so that the cable as a whole is electrically neutral. Find the potential difference between a point on the axis and a point on the outer cylinder.

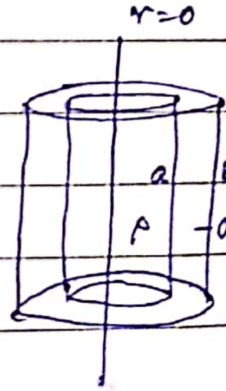
ans:- Note from problem 2.16, the electric field is,

$$\vec{E} = \frac{\rho r}{2\varepsilon_0}\hat{r} \quad \text{for } r<a$$

$$= \frac{\rho a^2}{2\varepsilon_0 r}\hat{r} \quad \text{for } a<r<b$$

$$=0 \quad \text{for } r>b.$$

For any reference point in the outer surface of the coaxial cable $r=b$.



$$\therefore V = V(b)-V(0) = -\int_0^b \vec{E}\cdot\vec{dl}$$

$$= -\int_0^a \vec{E}\cdot\vec{dl} - \int_a^b \vec{E}\cdot\vec{dl}$$

$$= -\frac{\rho}{2\varepsilon_0}\cdot\int_0^a r\,dr - \frac{\rho a^2}{2\varepsilon_0}\int_a^b \frac{dr}{r}$$

$$= \frac{-\rho}{2\varepsilon_0}\cdot\frac{r^2}{2}\Big|_0^a - \frac{\rho a^2}{2\varepsilon_0}\ln(r)\Big|_a^b$$

$$= -\frac{\rho a^2}{4\varepsilon_0}\left[1+2\ln\left(\frac{b}{a}\right)\right]$$

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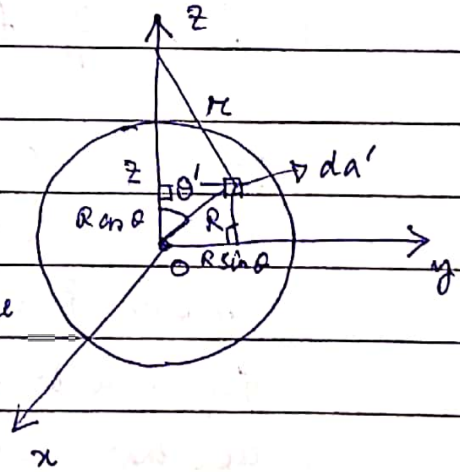
Poisson's equation and Laplace's equation

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{and} \quad \nabla \times \vec{E} = 0.$$

$\vec{E} = -\nabla V$. So, $-\nabla \cdot (\nabla V) = \frac{\rho}{\varepsilon_0} \Rightarrow \boxed{\nabla^2 V = -\frac{\rho}{\varepsilon_0}}$

This is Poisson's equation. In the region where there is no charge, $\rho = 0$ and Poisson's equation reduces to Laplace's equation – $\nabla^2 V = 0$.

Example 2.7 Find the potential of a uniformly charged spherical shell of radius R.



ans:- $V(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{\sigma \, da'}{r}$

Now, $r^2 = R^2 + z^2 - 2Rz\cos\theta'$

An element of surface area on this sphere is $R^2 \sin\theta' d\theta' d\varphi'$. So,

$$V(z) = \frac{\sigma}{4\pi\varepsilon_0} \cdot \int \frac{R^2 \sin\theta' d\theta' d\varphi'}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}}$$

$$= 2\pi R^2 \sigma \int_0^{\pi} \frac{\sin\theta' d\theta'}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}}$$

$$= 2\pi R^2 \sigma \left(\frac{1}{Rz} \sqrt{R^2 + z^2 - 2Rz\cos\theta'} \right) \Big|_0^{\pi}$$

$$= \frac{2\pi R\sigma}{z} \left(\sqrt{R^2 + z^2 + 2Rz} - \sqrt{R^2 + z^2 - 2Rz} \right)$$

$$= \frac{2\pi R\sigma}{z} \left(\sqrt{(R+z)^2} - \sqrt{(R-z)^2} \right)$$

At this stage we must be very careful to take the positive root. For point outside the sphere, z is greater than R, and hence $\sqrt{(R-z)^2} = z-R$; for points inside the sphere, $\sqrt{(R-z)^2} = R-z$.

Thus,

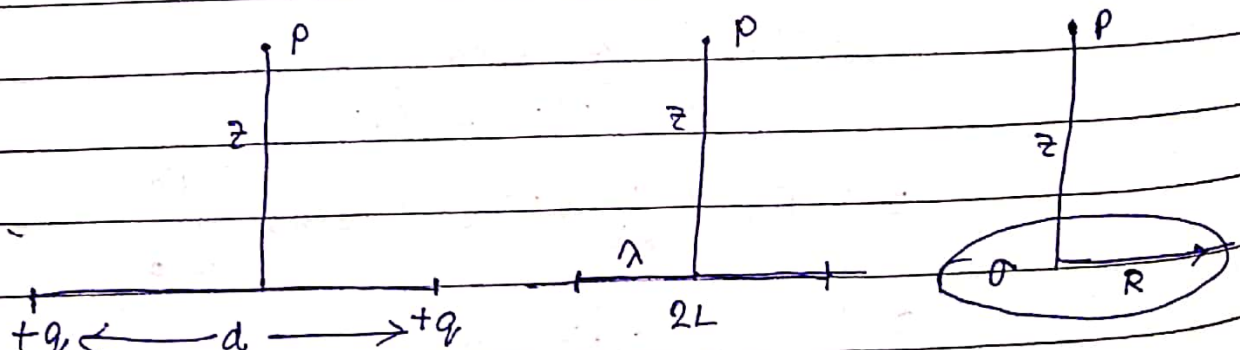
$$V(z) = \frac{R\sigma}{2\epsilon_0 z}\left[(R+z) - (z-R)\right] = \frac{R^2\sigma}{\epsilon_0 z} \quad \text{outside;}$$

$$V(z) = \frac{R\sigma}{2\epsilon_0 z}\left[(R+z) - (R-z)\right] = \frac{R\sigma}{\epsilon_0}, \quad \text{inside.}$$

In terms of the total charge on the shell, $q = 4\pi R^2\sigma$, $V(z) = \frac{1}{4\pi\epsilon_0}\cdot\frac{q}{z}$

[or in general, $V(r) = \frac{1}{4\pi\epsilon_0}\cdot\frac{q}{r}$] for points outside the sphere, and $\frac{1}{4\pi\epsilon_0}\cdot\frac{q}{R}$ for points ~~outside~~ inside.

Problem 2.25 Find the potential at a distance z above the centre of the charge distribution in the figures below. In each case, compute $\vec{E} = -\nabla V$; suppose we change the right hand charge in figure a to $-q$, what then is the potential at P? What field does that suggest?



(a) two point charge

(b) line charge.

(c) surface charge.

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ans:- (a) Refer to problem 2.2, $E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{2qz}{(z^2+d^2/4)^{3/2}}$

$$V = -\int_{\infty}^{z} \vec{E}\cdot\vec{dl} = -\int_{\infty}^{z} \frac{1}{4\pi\varepsilon_0} \cdot \frac{2qz}{(z^2+d^2/4)^{3/2}} \cdot dz$$

$$= -\frac{1}{4\pi\varepsilon_0} \cdot 2q \cdot \int_{\infty}^{z} \frac{z\,dz}{(z^2+d^2/4)^{3/2}}$$, Let, $z^2 + \frac{d^2}{4} = x^2$

$\cancel{2}z\,dz = \cancel{2}x\,dx$

$$= -\frac{1}{4\pi\varepsilon_0} \cdot 2q \int_{\infty}^{\sqrt{z^2+d^2/4}} \frac{\cancel{x}\,dx}{\cancel{x^3}\,x^2}$$

$$= -\frac{1}{4\pi\varepsilon_0} \cdot 2q \cdot \left[\frac{x^{-2+1}}{-2+1}\right]_{\infty}^{\sqrt{z^2+d^2/4}}$$

$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{2q}{\sqrt{z^2+d^2/4}}$$

(b) $$V = \frac{1}{4\pi\varepsilon_0} \cdot \int_{-L}^{+L} \frac{\lambda\,dx}{\sqrt{z^2+x^2}} = \frac{\lambda}{4\pi\varepsilon_0} \ln\left(x+\sqrt{z^2+x^2}\right)\Big|_{-L}^{L}$$

$$= \frac{\lambda}{4\pi\varepsilon_0} \cdot \ln\left[\frac{L+\sqrt{L^2+z^2}}{-L+\sqrt{L^2+z^2}}\right]$$

(c) $$V = \frac{1}{4\pi\varepsilon_0} \int_{0}^{R} \frac{\sigma\cdot 2\pi r\,dr}{\sqrt{r^2+z^2}} = \frac{1}{4\pi\varepsilon_0} \cdot 2\pi\sigma \left(\sqrt{z^2+r^2}\right)\Big|_{0}^{R}$$

$$= \frac{1}{4\pi\varepsilon_0} \cdot 2\pi\sigma \cdot \left(\sqrt{R^2+z^2} - z\right).$$

(a) $$\vec{E} = -\frac{dV}{dz} = -\frac{1}{4\pi\varepsilon_0} \cdot 2q \cdot \left(-\frac{1}{2}\right) \cdot \frac{2z}{(z^2+d^2/4)^{3/2}}\,\hat{z}$$

$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{2qz}{\{z^2+d^2/4\}^{3/2}}\,\hat{z}$$

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(b) $\vec{E} = -\frac{\lambda}{4\pi\varepsilon_0} \cdot \left\{ \frac{1}{L+\sqrt{z^2+L^2}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{z^2+L^2}} \cdot 2z - \frac{1}{(-L+\sqrt{z^2+L^2})} \cdot \frac{1}{2} \cdot \frac{2z}{\sqrt{z^2+L^2}} \right\} \hat{z}$

$$= -\frac{\lambda}{4\pi\varepsilon_0} \cdot \frac{z}{\sqrt{z^2+L^2}} \left\{ \frac{-L+\sqrt{z^2+L^2}-L-\sqrt{z^2+L^2}}{(z^2+L^2)-L^2} \right\} \hat{z}$$

$$= \frac{2\lambda L}{4\pi\varepsilon_0} \cdot \frac{1}{z\sqrt{z^2+L^2}} \hat{z}.$$

(c) $\vec{E} = -\frac{\sigma}{2\varepsilon_0} \cdot \left\{ \frac{1}{2} \cdot \frac{1}{\sqrt{R^2+z^2}} \cdot 2z - 1 \right\} \hat{z} = \frac{\sigma}{2\varepsilon_0} \cdot \left[1 - \frac{z}{\sqrt{R^2+z^2}} \right] \hat{z}$

In (a), if the right-hand charge is $-q$, then $V=0$, which, suggests that $\vec{E} = -\nabla V = 0$, in contradiction with previous calculations. The point is that we only know V on the z-axis, and from this we cannot hope to compute $E_x = -\frac{dV}{dx}$ or, $E_y = -\frac{dV}{dy}$. That was ok in part (a), because we know from symmetry that $E_x = E_y = 0$, But now $\vec{E}$ points in the x-direction, so knowing V on the x-axis is insufficient to determine $\vec{E}$.

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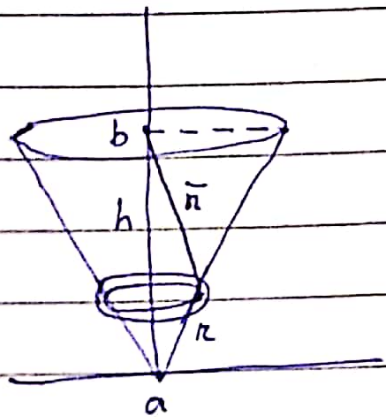
Problem 2.26 A conical surface (an empty ice-cream cone) carries a uniform surface charge σ. The height of the cone is h, as is the radius of the top. Find the potential difference between points $\vec{a}$ (the vertex) and $\vec{b}$ (the centre of the top).



ans:-

$$V(a) = \frac{1}{4\pi\varepsilon_0} \int_0^{\sqrt{2}h} \frac{\sigma \cdot 2\pi r \, d\mathcal{R}}{\mathcal{R}}$$

$$= \frac{2\pi\sigma}{4\pi\varepsilon_0} \cdot \frac{1}{\sqrt{2}} (\sqrt{2}h) = \frac{\sigma h}{2\varepsilon_0}.$$

where $r = \mathcal{R}/\sqrt{2}$.

$$V(b) = \frac{1}{4\pi\varepsilon_0} \cdot \int_0^{\sqrt{2}h} \frac{\sigma \cdot 2\pi r \, d\mathcal{R}}{\bar{\mathcal{R}}} \quad \text{where } \bar{\mathcal{R}} = \sqrt{h^2+\mathcal{R}^2-\sqrt{2}h\mathcal{R}}$$

$$= \frac{2\pi\sigma}{4\pi\varepsilon_0} \cdot \frac{1}{\sqrt{2}} \int_0^{\sqrt{2}h} \frac{\mathcal{R}}{\sqrt{h^2+\mathcal{R}^2-\sqrt{2}h\mathcal{R}}} \, d\mathcal{R}$$

$$= \frac{\sigma}{2\sqrt{2}\varepsilon_0} \left[\sqrt{h^2+\mathcal{R}^2-\sqrt{2}h\mathcal{R}} + \frac{h}{\sqrt{2}} \ln\left(2\sqrt{h^2+\mathcal{R}^2-\sqrt{2}h\mathcal{R}} + 2\mathcal{R} - \sqrt{2}h\right) \right]_0^{\sqrt{2}h}$$

$$= \frac{\sigma}{2\sqrt{2}\varepsilon_0} \cdot \left[h + \frac{h}{\sqrt{2}} \ln(2h + 2\sqrt{2}h - \sqrt{2}h) - h - \frac{h}{\sqrt{2}} \ln(2h - \sqrt{2}h) \right]$$

$$= \frac{\sigma}{2\sqrt{2}\varepsilon_0} \cdot \frac{h}{\sqrt{2}} \left[\ln(2h+\sqrt{2}h) - \ln(2h-\sqrt{2}h) \right]$$

$$= \frac{\sigma h}{4\varepsilon_0} \ln\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right) = \frac{\sigma h}{4\varepsilon_0} \ln\left(\frac{(2+\sqrt{2})^2}{2}\right)$$

$$= \frac{\sigma h}{2\varepsilon_0} \ln(1+\sqrt{2})$$

So, $V(a) - V(b) = \frac{\sigma h}{2\varepsilon_0} \left[1 - \ln(1+\sqrt{2}) \right]$