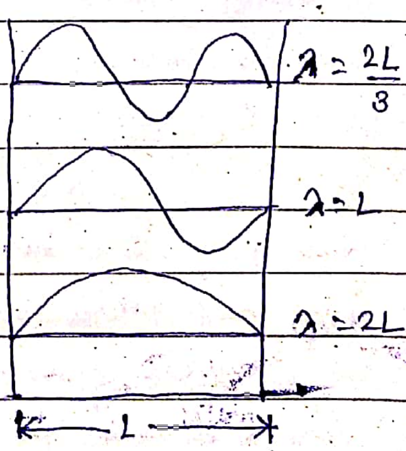
Modern Physics

Blackbody radiation – we are all familiar with the glow of a hot piece of metal. However, all objects neednot be so hot that it is luminous for it to be radiating e.m. energy, All objects radiate such energy continuously, whatever their temperature, though, which frequency they radiate depends on the temperature. The ability of a body to radiate is closely related to its ability to absorb radiation. This is because, a body at constant temperature is in thermal equilibrium with its surroundings and must absorb energy from them at the same rate as it emits energy. It is convinient to consider as an ideal body one that absorbs all radiation incident upon it, regardless of frequency. Such a body is called a black body.

In the laboratory, a black body can be approximated by a hollow ~~black~~ object with a very small hole leading to its interior. Any radiation striking the hole enters the cavity, where it is trapped by reflection back and forth until it is absorbed. The cavity walls are constantly emitting and absorbing radiation.

The ultraviolet catastrophe –

Lord Raleigh and James Jean studied the nature of blackbody radiation. In their study, they started by considering the radiation inside a cavity at absolute temperature T whose walls are perfect reflectors as – a series of standing e.m. waves. The condition for standing waves in such cavity is that the path length from wall to wall, whatever the direction, must be a whole number of half wavelength, so that, a node occurs at each reflecting surface. The number of independent standing waves $G(\nu)\,d\nu$ in the frequency interval ν and $\nu + d\nu$ per unit volume in the cavity turned out to be –



Density of standing waves in cavity – $G(\nu)\,d\nu = \dfrac{8\pi\nu^2\,d\nu}{c^3}$

This formula is independent of cavity shape. Now, to find the average energy per standing wave, we can consider each standing wave originating from the oscillation of electric charge inside the cavity wall. Now, a one-dimensional harmonic oscillator has two degrees of freedom, one corresponding to its kinetic energy and one corresponding to its potential energy. Hence, two degrees of freedom are associated with the standing waves, and, it should have an average energy of $2 \times \frac{1}{2}KT$. $\bar{\epsilon} = KT$. The $\frac{1}{2}KT$ originating from the fact that according to law of equipartition of energy, each degrees of freedom is associated with $\frac{1}{2}KT$ energy. The total energy $u(\nu)\,d\nu$ per unit volume in the cavity in the frequency range from ν to $\nu + d\nu$ is therefore,

Raleigh - Jeans formula – $u(\nu)\,d\nu = \bar{\epsilon}\, G(\nu)\,d\nu = \frac{8\pi KT}{c^3}\, \nu^2\, d\nu$.

From this formula it is evident that, as $\nu \to$ higher frequency towards the u.v. range of the spectrum, $u(\nu)$ should increase as ν^2. So, $\nu \to \infty$, $u(\nu) \to \infty$. However, in reality, the energy density (and radiation rate) falls to zero as $\nu \to \infty$. This discrepancy is known as <u>ultraviolet catastrophe</u>.

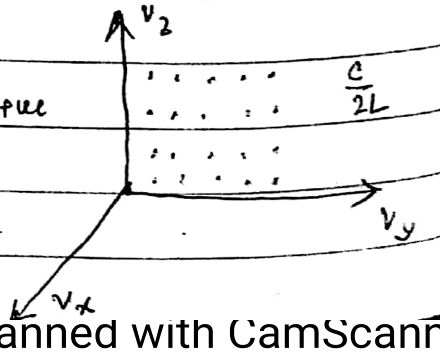
<u>Plank's radiation formula</u> –

In 1900, Max Plank came up with the formula for the energy density of black body radiation – $u(\nu)\,d\nu = \frac{8\pi h}{c^3} \cdot \frac{\nu^3\, d\nu}{e^{h\nu/KT} - 1}$

<u>Derivation</u> – To calculate the no. of modes of oscillation of the em waves inside the cavity, consider a one dimensional box of sides L. In equilibrium only standing waves are possible. The standing waves will have nodes at the ends $x=0$ and $x=L$. So, $L = n_x \cdot \frac{\lambda}{2}$ as discussed earlier. Since $\nu\lambda = c$, so,

$$\nu = n_x \cdot \frac{c}{2L}$$

There will be two modes for each trio of integers (n_x, n_y, n_z), since there are two independent polarisations possible. To find the numbers of modes with frequency between ν and $\nu + d\nu$, look at an array of points:



There is one point per cube of volume $\left(\frac{c}{2L}\right)^3$, and only +ve n_x, n_y, n_z are acceptable. Thus, the no. of triplets of (n_x, n_y, n_z) is equivalent to the volume of one octant of the space divided by volume $\left(\frac{c}{2L}\right)^3$.

$\therefore$ No. of modes of oscillation between ν and $\nu + d\nu$ $= \dfrac{2 \times \frac{1}{8} \times 4\pi\nu^2 d\nu}{(c/2L)^3}$

$$= \frac{8\pi}{c^3} \cdot V \cdot \nu^2 d\nu \quad \text{where } V = L^3.$$

The factor $4\pi\nu^2 d\nu$ is the volume of a thin spherical shell of radius ν and $\nu + d\nu$. and $V = L^3$ is the volume of the cubical cavity. Giving the density of states –

$$g(\nu) = \frac{8\pi V}{c^3} \cdot \nu^2 d\nu \quad \text{or,} \quad g(\lambda) = \frac{8\pi V}{\lambda^4} d\lambda.$$

for $2 \times \frac{1}{2} KT = KT$ energy / mode, the energy density could be expressed as –

$$u(\nu)\, d\nu = \frac{8\pi KT}{c^3} \nu^2 d\nu,$$ which is the Raleigh – Jeans formula leading to ultraviolet catastrophe.

Plank's new idea was to assume that the possible energies of the oscillators were quantized, ie, $E_n = nh\nu$, $n = 0, 1, 2, \ldots$ and $h =$ Plank's constant.

The average energy per oscillator was calculated using M–B Statistics –

$$\bar{\epsilon} = \frac{\sum_n E_n e^{-E_n/KT}}{\sum_n e^{-E_n/KT}}, \quad Z = \sum e^{-E_n/KT} \text{ is the partition function.}$$

$$Z = \sum_{n=0}^{\infty} e^{-E_n/KT} = \sum_{n=0}^{\infty} e^{-nx} = \frac{1}{1-e^{-x}} \quad \text{where } x = \frac{h\nu}{KT}.$$

and $$\sum_{n=0}^{\infty} E_n e^{-E_n/KT} = \sum_{n=0}^{\infty} nh\nu\, e^{-nx} = h\nu\left(-\frac{dZ}{dx}\right) = \frac{h\nu e^{-x}}{(1-e^{-x})^2}$$

So, the avg. energy / oscillator $= \bar{\epsilon} = \dfrac{h\nu}{e^x - 1} = \dfrac{h\nu}{e^{h\nu/KT} - 1}$.

Thus, $$\boxed{u(\nu)\, d\nu = \frac{8\pi}{c^3} \cdot \bar{\epsilon} \cdot \nu^2 d\nu = \frac{8\pi}{c^3} \cdot \frac{h\nu^3 d\nu}{e^{h\nu/KT} - 1}},$$

and $$\boxed{u(\lambda)\, d\lambda = \frac{8\pi hc}{\lambda^5} \cdot \frac{1}{e^{hc/\lambda KT} - 1} d\lambda.}$$

Then, the total energy, $u = \int_0^{\infty} u(\nu)\, d\nu$ per unit volume.

$\therefore$, $u(T) = \frac{8\pi h}{c^3} \int_0^\infty \frac{\nu^3 d\nu}{e^{h\nu/KT}-1} = \frac{8\pi (KT)^4}{(hc)^3} \int_0^\infty \frac{x^3}{e^x-1} dx = \frac{\pi^4}{15}$.

So, $\boxed{u(T) = \frac{8\pi^5 (KT)^4}{15(hc)^3} \propto T^4}$ $\rightarrow$ a form of Stefan - Boltzmann law.

Now, the flux of radiation is calculated by integrating over all frequencies, the expression $F_\nu \, d\nu = \frac{\text{energy}}{\text{area. time}}$ with frequency between ν and $\nu + d\nu$, and is given by, $F = \frac{c}{4} \cdot u(T) = \frac{ac}{4} T^4 = \sigma T^4$

$\therefore$, $\boxed{F = \sigma T^4}$ $\rightarrow$ Stefan - Boltzmann law,

$\sigma = 5.670 \times 10^{-8} \, W/m^2K^4 = 5.670 \times 10^{-5} \, erg/cm^2 s K^4$ $\rightarrow$ Stefan's constant.

In the limits $\nu \to 0$, low frequency, $e^x \approx 1+x$. (ie, for $h\nu/KT \ll 1$)

we have, $u(\nu)\,d\nu = \frac{8\pi h}{c^3} \cdot \nu^3 \cdot \frac{d\nu}{\frac{h\nu}{KT} + \cancel{1} - \cancel{1}} = \frac{8\pi \cancel{h}}{c^3} \cdot \frac{KT}{\cancel{h\nu}} \cdot \cancel{\nu^3}\,\nu^2 \, d\nu$

or, $\boxed{u(\nu)\,d\nu = \frac{8\pi}{c^3} (KT) \cdot \nu^2 d\nu}$ $\rightarrow$ Raleigh - Jean formula

In this formula it is worthwhile to note that the Plank's constant h is absent. This is an example of the correspondance principle, as h becomes negligible compared to other quantities in quantum mechanical law, the result approaches the classical law.

In the limits $\nu \to \infty$, high frequency, $h\nu \gg KT$ and $e^{h\nu/KT} \to \infty$

Then, $u(\nu)\,d\nu = \frac{8\pi h}{c^3} \cdot \nu^3 \cdot e^{-h\nu/KT}$

as $e^{h\nu/KT} \to \infty$, $u(\nu)\,d\nu \to 0$, So ultraviolet catastrophe is avoided.

The frequency or wavelength of maximum flux can be found by setting $\frac{dF_\nu}{d\nu} = 0$.

Since, $F_\nu = \frac{c}{4} \cdot u_\nu$ so,

$$\frac{c}{4} \frac{d}{d\nu} \left[\frac{8\pi h}{c^3} \cdot \frac{\nu^3}{e^{h\nu/KT}-1} \right] = 0.$$

$$\Rightarrow \frac{c}{4} \cdot \frac{8\pi h}{c^3} \cdot \left[\frac{3\nu^2}{e^{h\nu/KT}-1} - \frac{\nu^3 \cdot (h/KT) \cdot e^{h\nu/KT}}{(e^{h\nu/KT}-1)^2} \right] = 0$$

$$\Rightarrow \frac{2\pi h}{c^2} \cdot \frac{1}{(e^{h\nu/KT}-1)} \cdot \left[3\nu^2 - \frac{\nu^3 \cdot (h/KT) \cdot e^{h\nu/KT}}{(e^{h\nu/KT}-1)} \right] = 0$$

taking $x = h\nu/KT$, this reduces to,

$\frac{xe^x}{e^x-1} = 3$, which is a transcendental equation, The solution is, $x = 2.821$ ∴ or, $\boxed{\nu_{max} = \frac{2.821\, KT}{h}}$

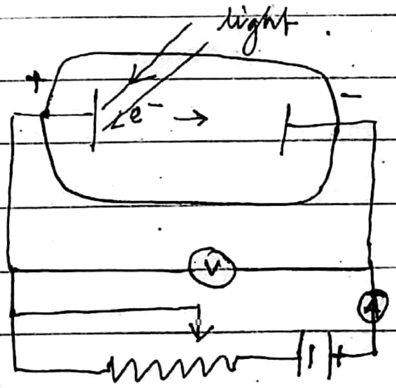
Solving for wavelength, $0 = \frac{dF_\lambda}{d\lambda} \Rightarrow \frac{ye^y}{e^y-1} = 5$, where, $y = \frac{hc}{\lambda KT}$ and gives $y = 4.965$.

$$\boxed{\lambda_{max} = \frac{hc}{4.965\,KT} = \frac{2.90\ mm.K}{T}} \rightarrow \text{Wein's displacement law,}$$

So, at room temperature of say 290K, $\lambda_{max} = 0.01\,mm = 10\,\mu m$, in the infrared, ie, the flux is maximum at $10\,\mu m$ wavelength. But our eyes cannot detect it.

Photoelectric effect —

An evacuated tube contains two electrodes connected to a source of variable voltage, with the metal plate whose surface is irradiated as the anode. Some of the photo electrons that emerge from this surface have enough energy to reach the cathode despite its -ve polarity, and they constitute the measured current. The slower photoelectrons are repelled before they reach the cathode surface. When the voltage is increased to a certain value V_0, of the order of several volts, no more photoelectrons arrive, as indicated by current dropping to zero. This extinction voltage corresponds to the maximum photoelectron K.E.



experimental observation of photoelectric effect.

This emission of e^- from the metal surface upon incidence of light above a certain frequency is called photoelectric effect, and the emitted electrons are called photoelectrons. The following characteristics of photoelectric effect were observed-

1. Within the limits of experimental accuracy (about $10^{-9}s$), there is no time interval between the arrival of light at the metal surface and the emission of photoelectrons. However, because the energy in an e.m. wave is supposed to be spread across the wavefronts, a period of time should elapse before an individual e^- accumulates enough energy (several eV) to leave the metal.
2. A bright light yeilds more photoelectrons than a dim one of the same frequency, but the electron energies remain the same. The last part is contrary to em theory.
3. The higher the frequency of light, the more energy the photoelectrons have. Blue light results in a faster e^- than red light.
4. At frequencies below a certain critical frequency ν_0, which is characteristic of each particular metal, no e^- are emitted. Above ν_0, the photoelectrons range in energy from zero to maximum value that increases linearly with increasing frequency. This observation cannot be explained by em theory of light.

To get over these dilemas, Einstein realized that the photoelectric effect could be understood if the energy in light is not spread out over wavefronts but is concentrated in small packets, or photons. Each photon of light of frequency ν has the energy $h\nu$, the same as Plank's quantum energy.

Plank had thought that, although energy from an electric oscillator apparently had to be given to em waves in separate quanta of $h\nu$ each, the waves themselves behaved exactly as in conventional wave theory. Einstein's break with classical physics was more drastic: Energy was not only given to em waves in separate quanta of $h\nu$ each but was also carried by the waves in separate quanta.

The experimental observations listed above directly follows from Einstein's hypothesis —

(1) Because em wave energy is concentrated in photons and not spread out, there should be no delay in emission of photoelectrons.

(2) All photons of frequency ν have the same energy, so changing the

intensity of the monochromatic light will change the number of photoelectrons but not their energy.

(3) & (4) The higher the frequency ν, the greater the photon energy $h\nu$ and so the more energy the photoelectrons have. To explain the ~~minimum energy of~~ critical frequency ν_0, below which no photoelectrons are emitted, there must be a minimum energy φ for an e^- to escape from a particular metal surface or else e^- would pour out all the time. This energy is called the work function of the metal and is related to ν_0 as, $\varphi = h\nu_0$. The greater the work function of a metal, the more energy is needed for an e^- to leave its surface, and higher the critical frequency for photoelectric emission to occur.

According to Einstein, the photoelectric effect in a given metal should obey the equation – $\boxed{h\nu = KE_{max} + \varphi}$, where $h\nu$ is the photon energy, KE_{max} is the maximum photoelectron energy (which is proportional to the stopping potential), and φ the minimum energy needed for an e^- to leave the metal surface. Because, $\varphi = h\nu_0$, $\boxed{KE_{max} = h(\nu - \nu_0)}$ → This indeed is a linear equation as observed experimentally.

<u>Example</u> – ultraviolet light of wavelength 350 nm and intensity $1.00\ W/m^2$ is directed at a potassium surface. (a) Find the maximum KE of the photoelectrons. (b) If 0.50 % of the incident light photons produce photoelectrons, how many are emitted per second if the potassium surface has an area of $1.00\ cm^2$?

<u>ans</u>:- (a) energy of the incident photon, $E_p = \dfrac{1240\ (eV)}{\lambda(nm)} = \dfrac{1240}{350} = 3.54\ eV$.

work function of potassium $= 2.2\ eV$. So, $KE_{max} = (3.54 - 2.20) = 1.3\ eV$.

(b) The photon energy in Joule is 5.68×10^{-19} J. Hence the no. of photons that reach the surface per second is $n_p = \dfrac{E/t}{E_p} = \dfrac{(1.00\ W/m^2)(1.00 \times 10^{-4}\ m^2)}{5.68 \times 10^{-19}\ J/photon}$

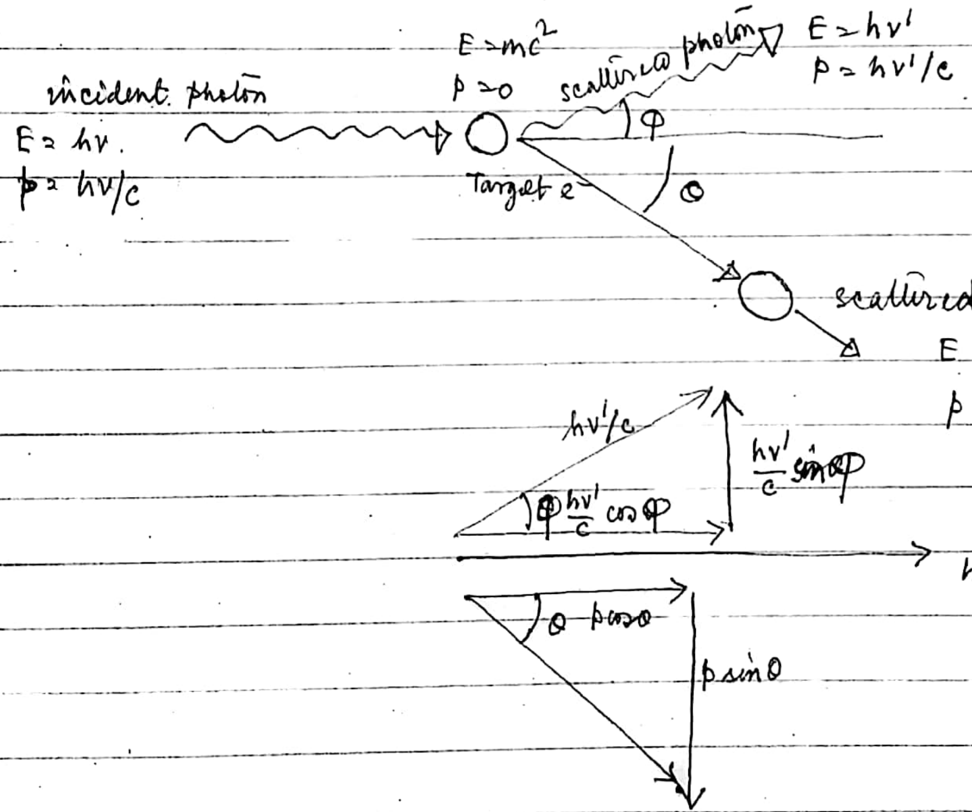
$= 1.76 \times 10^{14}$ photons/s.

The rate at which photoelectrons are emitted are therefore, $n_e = (0.05)\,n_p =$ 8.8×10^{11} photo e^-/s.

Compton effect –

According to quantum theory of light, photons behave as particles except for their lack of rest mass. Figure below shows such a collision : an x-ray photon strikes an e^- (assumed to be initially at rest in the laboratory frame) and is scattered away from its original direction of motion while the e^- receives an impulse and begins to move. We can think of the photon as loosing an amount of energy in the collision that is the same as the kinetic energy KE gained by the e^-, although actually seperate photons are involved. If the initial photon has frequency ν, the scattered photon has lower frequency ν', where –

loss in photon energy = gain in e^- energy.



The initial photon momentum is $h\nu/c$ and the scattered photon momentum is $h\nu'/c$, and the initial and final e^- momenta are respectively 0 and p. In the original photon direction – initial momentum = final momentum.

$$\frac{h\nu}{c} + 0 = \frac{h\nu'}{c}\cos\varphi + p\cos\theta \quad \text{—} (1)$$

In $\perp r$ direction, initial momentum = final momentum

$$0 = \frac{h\nu'}{c}\sin\varphi - p\sin\theta \quad \text{—} (2)$$

The angle φ is that between the directions of the initial and scattered photons, and θ is that between the directions of the initial photon and the recoil e^-.

Now, multiplying ① & ② with c, we can write

$$h\nu = h\nu' \cos\varphi + pc\cos\theta \quad \text{— ③}$$

$$pc\sin\theta = h\nu'\sin\varphi \quad \text{— ④}$$

squaring and adding,

$$p^2c^2\sin^2\theta + p^2c^2\cos^2\theta = (h\nu)^2 + (h\nu')^2\cos^2\varphi - 2(h\nu)(h\nu')\cos\varphi + (h\nu')^2\sin^2\theta$$

$$\Rightarrow \quad p^2c^2 = (h\nu)^2 + (h\nu')^2 - 2(h\nu)(h\nu')\cos\varphi. \quad \text{— ⑤}$$

Now, $E^2 = p^2c^2 + m^2c^4 \Rightarrow (KE + mc^2)^2 = m^2c^4 + p^2c^2$

$$\Rightarrow (KE)^2 + \cancel{m^2c^4} + 2(KE)(mc^2) = \cancel{m^2c^4} + p^2c^2$$

$$\Rightarrow \quad p^2c^2 = (KE)^2 + 2(KE)(mc^2)$$

Since $KE = h\nu - h\nu'$, hence, $p^2c^2 = (h\nu - h\nu')^2 + 2(mc^2)(h\nu - h\nu')$

$$\Rightarrow p^2c^2 = (h\nu)^2 + (h\nu')^2 - 2(h\nu)(h\nu') + 2(mc^2)(h\nu - h\nu') \quad \text{— ⑥}.$$

Substituting in ⑤, $\cancel{(h\nu)^2} + \cancel{(h\nu')^2} - \cancel{2}(h\nu)(h\nu') + \cancel{2}(mc^2)(h\nu - h\nu') =$
$\cancel{(h\nu)^2} + \cancel{(h\nu')^2} - \cancel{2}(h\nu)(h\nu')\cos\varphi$.

$$\Rightarrow \quad -(h\nu)(h\nu') + (h\nu)(h\nu')\cos\varphi = (mc^2)(h\nu' - h\nu)$$

$$\Rightarrow \quad (h\nu)(h\nu')(1-\cos\varphi) = (mc^2)(h\nu - h\nu'). \quad \text{— ⑦}$$

$$\Rightarrow \quad \left(\frac{hc}{\lambda}\right)\left(\frac{hc}{\lambda'}\right)(1-\cos\varphi) = (mc^2)\left(\frac{hc}{\lambda} - \frac{hc}{\lambda'}\right)$$

$$\Rightarrow \quad \frac{hc}{\lambda\lambda'}(1-\cos\varphi) = mc^2\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)$$

$$\Rightarrow \quad \frac{hc}{\cancel{\lambda\lambda'}}(1-\cos\varphi) = \frac{mc^2(\lambda' - \lambda)}{\cancel{\lambda\lambda'}}$$

$$\Rightarrow \quad \boxed{(\lambda' - \lambda) = \frac{h}{mc}(1-\cos\varphi)} \rightarrow \text{Compton Effect.}$$

This eqn. gives the change in wavelength expected for a photon that is scattered through the angle φ by a particle of rest mass m. The change is independent of the wavelength λ of the incident photon. $\lambda_c = h/mc$ is called the Compton wavelength of the scattering particle. For an e^-, $\lambda_c = 2.426 \times 10^{-12}$ m. In terms of λ_c,

$$\boxed{\Delta\lambda = \lambda' - \lambda = \lambda_c (1 - \cos\varphi)} \quad \text{—} (7).$$

The compton effect is the chief means by which x-rays lose energy when passing through matter.

Example x-rays of wavelength 10 pm are scattered from a target. (a) find the wavelength of the x-rays scattered through 45°. (b) find the maximum wavelength present in the scattered x-rays. (c) Find the maximum kinetic energy of the recoil electrons. ($\lambda_c = 2.426 \times 10^{-12}$ m)

ans :— a) $\lambda' - \lambda = \lambda_c(1 - \cos\varphi)$, $\lambda' = \lambda + \lambda_c(1 - \cos\varphi)$

$$\Rightarrow \lambda' = 10.0 \text{ pm} + \lambda_c(1 - \cos 45°)$$
$$= 10.0 \text{ pm} + 0.293\,\lambda_c$$
$$= 10.7 \text{ pm}.$$

b) $\lambda' - \lambda$ is maximum when $1 - \cos\varphi = 2$, in which case

$$\lambda' = \lambda + 2\lambda_c = 10.0 \text{ pm} + 4.9 \text{ pm} = 14.9 \text{ pm}.$$

c) The maximum recoil kinetic energy is equal to the difference between the energies of the incident and scattered photons, so,

$$KE_{max} = h(\nu - \nu') = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)$$

$$\therefore KE_{max} = \frac{(6.625 \times 10^{-34} \text{ J s})(3 \times 10^8 \text{ m/s})}{10^{-12} \text{ m/pm}} \left(\frac{1}{10.0 \text{ pm}} - \frac{1}{14.9 \text{ pm}}\right)$$

$$= 6.54 \times 10^{-15} \text{ J}. = 40.8 \text{ keV}.$$

Problems -

1. If Plank's constant were smaller than it is, would quantum phenomenon be more or less conspicuous than they are now?

ans:- Plank's constant gives a measure of the energy at which quantum effects are observed. If Plank's constant had a smaller value, while all other physical quantities, such as speed of light, remained the same, quantum effects would be seen for phenomena that occur at higher energy scale or shorter wavelengths, ie, quantum phenomena would be less conspicuous than they are now.

2. Is it correct to say that the maximum photoelectron energy KE_{max} is proportional to the frequency ν of the incident light? if not, what would be the correct statement of relationship between KE_{max} and ν?

ans:- No. The relation is given by $KE_{max} = h\nu - \varphi = h(\nu - \nu_0)$, so while the KE_{max} is a linear function of the frequency ν and the incident light, it is not proportional to the frequency

3. Find the energy of a 700 nm photon.

ans:- $E = \frac{1240}{\lambda\,(nm)}\ eV = \frac{1240}{700} = 1.77\ eV.$

in Joules, $E = \frac{(6.63 \times 10^{-34}\ J.s)(3.0 \times 10^{8}\ m/s)}{700\ nm} = 2.84 \times 10^{-19}\ J.$

4. A 1.00-KW radio transmitter operates at a frequency of 880 kHz. How many photons per second does it emit?

ans:- The number of photons per unit time is the total energy per unit time (the power) divided by the energy per photon, or

$$\frac{P}{E} = \frac{P}{h\nu} = \frac{1.00 \times 10^3\ J/s}{(6.63 \times 10^{-34}\ J.s)(880 \times 10^3\ Hz)} = 1.72 \times 10^{30}\ photons/s.$$

5. Light from the sun arrives at the earth, an average of 1.5×10^{11} m away, at the rate of $1.4 \times 10^3\ W/m^2$ of area perpendicular to the direction of the light. Assume that sunlight is monochromatic with a frequency of 5.0×10^{14} Hz. (a) How many photons fall per second on each square meter of the earth's surface directly facing the sun? (b) What is the power output of the sun, and how many photons per second does it emit? (c) How many photons per cubic meter are there near the earth?

Ans:- (a) The number of photons per unit time per unit area (the power per unit area, P/A), divided by the energy per photon, or

$$\frac{P/A}{h\nu} = \frac{1.4 \times 10^3\ W/m^2}{(6.63 \times 10^{-34}\ J.s)(5.0 \times 10^{14}\ Hz)} = 4.2 \times 10^{21}\ photons/(s \cdot m^2)$$

(b) With the reasonable assumption that the sun radiates uniformly in all directions, all points at the same distance from the sun should have the same flux of energy, even if there is no surface to absorb the energy. The total power is then,

$$(P/A) 4\pi R_{E-S}^2 = (1.4 \times 10^3\ W/m^2)\, 4\pi\, (1.5 \times 10^{11}\ m)^2 = 4.0 \times 10^{26}\ W,$$

where R_{E-S} is the mean Earth - Sun distance, commonly abbreviated as "1 AU," for "astronomical unit." The number of photons emitted per second is this power divided by the energy per photon, or ~~unit~~

$$\frac{4.0 \times 10^{26}\ J/s}{(6.63 \times 10^{-34}\ J.s)(5.0 \times 10^{14}\ Hz)} = 1.2 \times 10^{45}\ photons/s$$

6. (c) The photons are all moving at the same speed c, and in the same direction (spreading is not significant on the scale of the earth), and so the number of photons per unit time per unit area is the product of the number per unit volume and the speed. Using the result from part (a),

$$\frac{4.2 \times 10^{21}\ photons/(s.m^2)}{3.0 \times 10^8\ m/s} = 1.4 \times 10^{13}\ photons/m^3.$$

6. The maximum wavelength of light that will cause photoelectrons to be emitted from sodium? What will the maximum kinetic energy of the photoelectrons be if 200-nm light falls on a sodium surface?

ans:- The maximum wavelength would correspond to the least energy that would allow an electron to be emitted, so the incident energy would be equal to work function, and

$$\lambda_{max} = \frac{hc}{\phi} = \frac{1.24 \times 10^{-6}\ eV.m}{2.3\ eV} = 539\ m$$

where the value of ϕ for sodium is taken from Table 2.1.

$$K_{max} = h\nu - \phi = \frac{hc}{\lambda} - \phi = \frac{1.24 \times 10^{-6}\ eV \cdot m}{200 \times 10^{-9}\ m} - 2.3\ eV = 3.9\ eV.$$

7. 1.5 mV of 400-nm light is directed at a photoelectric cell. If 0.10 percent of the incident photons produce photoeletrons, find the current is the cell.

ans :- Because only 0.10% of the light creates photoelectrons, the available power is $(1.0 \times 10^{-3})(1.5 \times 10^{-3}\ W) = 1.5 \times 10^{-6}\ W$. The current will be the product of the number of photoelectrons per unit time and the electrons charge, or

$$I = e\frac{P}{E} = e\frac{P}{hc/\lambda} = e\frac{P\lambda}{hc} = (1e)\frac{(1.5 \times 10^{-6}\ J/s)(400 \times 10^{-9}\ m)}{1.24 \times 10^{-6}\ eV \cdot m} = 0.48\mu A$$

8. A metal surface illuminated by 8.5×10^{14} Hz light emits electrons whose maximum energy is 0.52 eV. The same surface illuminated by 12.0×10^{14} Hz light emits electrons whose maximum energy is 1.97 eV. From there data find Planck's constant and the work function of the surface.

ans :- Denoting the two energies and frequencies with subcripts 1 and 2,

$$K_{max,1} = h\nu_1 - \phi, \quad K_{max,2} = h\nu_2 - \phi.$$

Subtracting to eliminate the work function ϕ and dividing by $\nu_1 - \nu_2$,

$$h = \frac{K_{max,2} - K_{max,1}}{\nu_2 - \nu_1} = \frac{19.7\ eV - 0.52\ eV}{12.0 \times 10^{14}\ Hz - 8.5 \times 10^{14}\ Hz} = 4.1 \times 10^{-15}\ eV\cdot s$$

to the allowed two significant figures. Keeping an extra figure gives

$$h = 4.14 \times 10^{-15}\ eV.s = 6.64 \times 10^{-34}\ J.s$$

The work function ϕ may be obtained by substituting the above result into either of the above expressions relating the frequencies and the energies, yielding $\phi = 3.0\ eV$ to the same two significant figures, or the equations may be solved by rewriting them as

$$K_{max,1} \nu_2 = h\nu_1\nu_2 - \phi\nu_2\ ,\ K_{max,2}\nu_1 = h\nu_2\nu_1 - \phi\nu_1\ ,$$

subtracting to eliminate the product $h\nu_1\nu_2$ and dividing by $\nu_1 - \nu_2$ to obtain

$$\phi = \frac{K_{max,2}\nu_1 - K_{max,1}\nu_2}{\nu_2 - \nu_1} = \frac{(19.7\,eV)(8.5\times10^{14}\,Hz) - (0.52\,eV)(12.0\times10^{14}\,Hz)}{(12.0\times10^{14}\,Hz - 8.5\times10^{14}\,Hz)} = 3.0\,eV$$

(This last calculation, while possibly more cumbersome than direct substitution, reflects the result of solving the system of equations using a symbolic-manipulation program; using such a program for this problem is, of course, a case of "swatting a fly with a sledgehammer".)

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9. Show that it is impossible for a photon to give up all its energy and momentum to a free electron. This is the reason why the photoelectric effect can take place only when photons strike bound electron.

ans:- Consider the proposed interaction in the frame of the e^- initially at rest. The photon's initial momentum is $p_0 = E_0/c$, and if the e^- were to attain all of the photon's momentum and energy, the final momentum of the e^- must be $p_e = p_0 = p$, the final electron kinetic energy must be $KE = E_0 = pc$, and so the final e^- energy is $E_e = pc + m_ec^2$ —①

However, for any e^- we must have $E_e^2 = p^2c^2 + m_e^2c^4$. —②

From ① & ②, $(pc + m_ec^2)^2 = p^2c^2 + m_e^2c^4 \Rightarrow 2(pc)(m_ec^2) = 0$.

This is only possible if $p = 0$, in which case the photon had no initial momentum and no initial energy, and hence could not have existed.

To see the same result without using as much algebra, the e^- final KE is

$$\sqrt{p^2c^2 + m_e^2c^4} - m_ec^2 \neq pc \qquad \text{for } p \neq 0.$$

An easier alternative is to consider the interaction in the frame where the e^- is at rest after absorbing the photon. In this frame, the final energy is the rest energy of the e^-, m_ec^2, but before the interaction, the e^- would have been moving (to conserve momentum), and hence would have had more energy than after the interaction, and the photon would have had positive energy, so energy could not be conserved.

10. A beam of x-ray is scattered by a target. At 45° from the beam direction the scattered x-rays have a wavelength of 2.2 pico.meter. What is the wavelength of the x-rays in the direct beam.

ans:- $\lambda' - \lambda = \lambda_c(1-\cos\varphi) \Rightarrow \lambda = \lambda' - \lambda_c(1-\cos\varphi) = 2.2\times10^{-12} - (2.426\times10^{-12})(1-\cos45°)$

$\therefore \lambda = 1.5\times10^{-12}\,m$.

11. At what scattering angle will incident 100 keV x-rays leave a target with an energy of 90 keV?

ans:- x-ray will undergo Compton scattering with the e^- inside the target and loose some energy. The shift in wavelength will be $\lambda' - \lambda = \lambda_c(1-\cos\varphi)$.
Now, $\nu\lambda = c$, or, $\lambda = \frac{c}{\nu}$, and $\lambda_c = \frac{h}{m_e c}$, where $m_e = m_{electron}$.

$$\therefore \quad 1-\cos\varphi = \frac{\lambda'-\lambda}{\lambda_c} \Rightarrow \cos\varphi = 1 - \frac{\lambda'}{\lambda_c} + \frac{\lambda}{\lambda_c}.$$

$$\Rightarrow \cos\varphi = 1 - \frac{\lambda'}{h/m_e c} + \frac{\lambda}{h/m_e c} \Rightarrow \cos\varphi = 1 - \frac{\lambda' c . m_e}{h} + \frac{\lambda c . m_e}{h}$$

$$\Rightarrow \cos\varphi = 1 - \frac{m_e c^2}{h\nu'} + \frac{m_e c^2}{h\nu} = 1 + m_e c^2\left(\frac{1}{h\nu} - \frac{1}{h\nu'}\right)$$

for e^-, $m_e c^2 = 511 keV$. so, $\cos\varphi = 1 + 511\ keV\left(\frac{1}{100} - \frac{1}{90}\right)\cdot\frac{1}{keV}$.

$$= 0.432$$

$$\therefore \quad \varphi = 64.4^\circ.$$

12. A photon of frequency ν is scattered by an electron initially at rest. verify that the maximum kinetic energy of the recoil electron is $KE_{max} = \frac{2h^2\nu^2}{mc^2} \Big/ \left(1 + \frac{2h\nu}{mc^2}\right)$

ans:- for the e^- to have maximum recoil energy, the scattering angle must be 180°.
$KE = h\nu - h\nu'$.

$$\lambda' - \lambda = \frac{h}{mc}(1-\cos\varphi).$$

$$\Rightarrow \frac{c}{\nu'} - \frac{c}{\nu} = \frac{h}{mc}(1-\cos\varphi) \Rightarrow \frac{c}{h\nu'} - \frac{c}{h\nu} = \frac{1}{mc}(1-\cos\varphi)$$

$$\Rightarrow \frac{mc^2}{h\nu'} - \frac{mc^2}{h\nu} = 1-\cos\varphi \Rightarrow mc^2\left(\frac{h\nu - h\nu'}{h\nu . h\nu'}\right) = 1-\cos\varphi.$$

Now, $KE_{max} = h\nu - h\nu'$, so, $mc^2 . \frac{KE_{max}}{(h\nu)(h\nu')} = 1 - \cos\varphi$.

for max scattering $\cos\varphi = -1$ i.e, $1+\cos\varphi = 2$

$$\Rightarrow \quad KE_{max} = \frac{2(h\nu)(h\nu')}{mc^2}$$

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Now, $\lambda' \nu' = \lambda \nu$, $\Rightarrow$ $\nu' = \frac{\lambda}{\lambda'} \nu$.

But, for max recoil, $\lambda' - \lambda = 2\lambda_c$ or, $\lambda' = \lambda + 2\lambda_c$

$$\Rightarrow \frac{1}{\lambda'} = \frac{1}{\lambda + 2\lambda_c} \Rightarrow \frac{\lambda}{\lambda'} = \frac{\lambda}{\lambda + 2\lambda_c} = \frac{1}{1 + 2\lambda_c/\lambda}$$

$$\therefore \nu' = \frac{\nu}{1 + 2\lambda_c/\lambda} \quad \text{or,} \quad h\nu' = \frac{h\nu}{1 + 2\lambda_c/\lambda}$$

But, $\lambda_c = \frac{h}{mc}$ or, $h\nu' = \frac{h\nu}{1 + 2h/mc\lambda} = \frac{h\nu}{1 + \frac{2h\nu}{mc^2}}$

So, $KE_{max} = \frac{2(h\nu).(h\nu)}{mc^2\left(1 + \frac{2h\nu}{mc^2}\right)} = \frac{2h^2\nu^2}{mc} \Big/ \left(1 + \frac{2h\nu}{mc^2}\right)$

13. Show that for Compton scattering the angles θ and φ are related as –

$$\tan\theta = \frac{1}{2}\cot\frac{\varphi}{2} \quad \text{and} \quad 2 - \cos\varphi = \frac{3 + 4\tan^2\theta}{1 + 4\tan^2\theta}$$

ans:- for compton scattering, from momentum conservation relations,

$$pc\cos\theta = h\nu - h\nu'\cos\varphi$$
$$pc\sin\theta = h\nu'\sin\varphi.$$

$\therefore \tan\theta = \frac{h\nu'\sin\varphi}{h\nu - h\nu'\cos\varphi}$, But $\Delta\lambda = \lambda_c(1 - \cos\varphi)$

or, $\lambda' - \lambda = \lambda_c(1 - \cos\varphi)$

or, $\lambda' - \lambda = \Delta\lambda \Rightarrow \lambda' = \lambda + \Delta\lambda \Rightarrow \frac{1}{\lambda'} = \frac{1}{\lambda + \Delta\lambda}$

$$\Rightarrow \frac{\lambda}{\lambda'} = \frac{\lambda}{\lambda + \Delta\lambda} = \frac{1}{1 + \frac{\Delta\lambda}{\lambda}}$$

Now, $\lambda'\nu' = \lambda\nu \Rightarrow \nu' = \frac{\lambda}{\lambda'}\nu = \frac{\nu}{1 + \frac{\Delta\lambda}{\lambda}}$

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$$\Rightarrow h\nu' = \frac{h\nu}{1+\Delta\lambda/\lambda} \Rightarrow h\nu = h\nu'\left(1+\frac{\Delta\lambda}{\lambda}\right)$$

$$\therefore \tan\theta = \frac{h\nu'\sin\varphi}{h\nu'\left(1+\frac{\Delta\lambda}{\lambda}\right)-h\nu'\cos\varphi} = \frac{\sin\varphi}{\left(1+\frac{\Delta\lambda}{\lambda}\right)-\cos\varphi} = \frac{\sin\varphi}{(1-\cos\varphi)+\frac{\Delta\lambda}{\lambda}}$$

$$= \frac{\sin\varphi}{(1-\cos\varphi)+\frac{\lambda_c}{\lambda}(1-\cos\varphi)} = \frac{\sin\varphi}{\left(1+\frac{\lambda_c}{\lambda}\right)(1-\cos\varphi)}$$

$$= \frac{\sin\varphi}{\left(1+\frac{h}{mc\lambda}\right)(1-\cos\varphi)} = \frac{\sin\varphi}{\left(1+\frac{hc}{\lambda}\cdot\frac{1}{mc^2}\right)(1-\cos\varphi)}$$

$$\left(\because \lambda_c = \frac{h}{mc}\right)$$

But $E = mc^2 = h\nu = \frac{hc}{\lambda}$, so,

$$\tan\theta = \frac{\sin\varphi}{2(1-\cos\varphi)} \quad -(1)$$

or, $$2\tan\theta = \frac{\sin\varphi}{1-\cos\varphi} \Rightarrow 4\tan^2\theta = \frac{\sin^2\varphi}{(1-\cos\varphi)^2}$$

$$\Rightarrow 4\tan^2\theta = \frac{1-\cos^2\varphi}{(1-\cos\varphi)^2} = \frac{(1-\cos\varphi)(1+\cos\varphi)}{(1-\cos\varphi)^2} = \frac{1+\cos\varphi}{1-\cos\varphi}$$

$$(\because \sin^2\varphi = 1-\cos^2\varphi).$$

$$\therefore (1-\cos\varphi)(4\tan^2\theta) = 1+\cos\varphi = 2-(1-\cos\varphi)$$

or, $$(1-\cos\varphi)(1+4\tan^2\theta) = 2 \Rightarrow 1-\cos\varphi = \frac{2}{1+4\tan^2\theta}$$

$$\Rightarrow 1+(1-\cos\varphi) = 1+\frac{2}{1+4\tan^2\theta} = \frac{1+4\tan^2\theta+2}{1+4\tan^2\theta} = \frac{3+4\tan^2\theta}{1+4\tan^2\theta}$$

$$\Rightarrow \boxed{2-\cos\varphi = \frac{3+4\tan^2\theta}{1+4\tan^2\theta}} \quad -(2)$$

Now, in ① using $\sin\varphi = 2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}$ and $1-\cos\varphi = 2\sin^2\frac{\varphi}{2}$,

$$\tan\theta = \frac{\cancel{2}\cancel{\sin}\frac{\varphi}{2}\cos\frac{\varphi}{2}}{2\cdot\cancel{2}\sin^{\cancel{2}}\frac{\varphi}{2}} = \frac{1}{2}\cot\frac{\varphi}{2}$$

ans $\boxed{\tan\theta = \frac{1}{2}\cot\frac{\varphi}{2}}$

14. Show that regardless the initial energy, a photon cannot undergo Compton scattering through an angle of more than 60° and still be able to produce an electron-positron pair.

ans:- $\lambda_c = \frac{h}{mc} = \frac{2hc}{2mc^2} = \frac{2hc}{E_{min}}$

Where $E_{min} = 2mc^2$ is the minimum photon energy needed for pair production. The scattered wavelength (a maximum) corresponding to this minimum energy is $\lambda'_{max} = ch/E_{min}$, so, $\lambda_c = 2\lambda'_{max}$. Here, it is possible to say that for the most energetic incoming photons, $\lambda \sim 0$, and so $1-\cos\varphi = \frac{1}{2}$ (considering $\nu \to$ very large, $\lambda \sim 1/\nu$ is very small), for $\lambda' = \lambda_c/2$, from which $\cos\varphi = \frac{1}{2}$ and $\varphi = 60°$. $\left(\because \lambda'_{max} - \lambda = \lambda_c(1-\cos\varphi) \right.$

$\lambda \to 0, \quad \lambda'_{max} = 2\lambda'_{max}(1-\cos\varphi)$

$\left. \Rightarrow 1-\cos\varphi = \frac{1}{2} \Rightarrow \cos\varphi = \frac{1}{2} \right)$