Heisenberg's uncertainty principle -

To regard a moving particle as a wave group implies that there are fundamental limits to the accuracy with which we can measure such particle properties as position and momentum. The Uncertainty Principle states that - It is impossible to know both the exact position and exact momentum of an object at the same time.

The de Broglie wavelength of a particle of momentum p is $\lambda = h/p$ and the corresponding wave number is — $k = \frac{2\pi}{\lambda} = \frac{2\pi p}{h}$

In terms of wave number the particles momentum is therefore — $p = \frac{hk}{2\pi}$.

Hence an uncertainty Δk in the wave number of the de Broglie waves associated with the particle results in an uncertainty Δp in the particle's momentum according to the formula — $\Delta p = \frac{h\Delta k}{2\pi}$

Since, $\Delta x \Delta k \geq \frac{1}{2}$, $\Delta k \geq 1/2\Delta x$ and $\Delta x \Delta p \geq h/4\pi$.

noting that $\hbar = h/2\pi$, $\Delta x \Delta p \geq \hbar/2$.

A particle approach to Heisenbergs uncertainty principle -

The uncertainty principle can be arrived at from the point of view of the particle properties of waves as well as from the point of view of the wave properties of particles. We might want to measure the position and momentum of an object at a certain ~~momentum~~. To do so, we must touch it with something that will carry the required information back to us. That is, we must poke it with a stick, shine a light on it, or perform some similar act. The measurement process itself thus requires that the object be interfered with in some way. If we consider such interference in details, we are led to the same uncertainty principle as before even without taking into account the wave nature of moving bodies.

Suppose we look at an electron using light of wavelength λ. Each photon of this light has the momentum h/λ. when one of these photons bounces off the electron (which must happen if we are to 'see' the electron), the electrons original momentum will be changed. The exact amount of the change Δp cannot be predicted, but it will be of the same order of magnitude as the photon momentum h/λ. Hence $\Delta p \approx h/\lambda$. — ①

The longer the wavelength of the observing photon, the smaller the uncertainty in the electron's momentum.

Because light is a wave phenomenon as well as a particle phenomenon, we cannot expect to determine the electron's location with perfect accuracy regardless of the instrument used. A reasonable estimate of the minimum uncertainty in the measurement might be one photon wavelength, so that $\Delta x \geq \lambda$. — ②

The shorter the wavelength, the smaller the uncertainty in location. However, if we use light of short wavelength to increase the accuracy of the position measurement, there will be a corresponding decrease in the accuracy of the momentum measurement because the higher photon momentum will disturb the electron's motion to a greater extent. Light of long wavelength will give a more accurate momentum but a less accurate position.

from ① & ②, $\Delta x \Delta p \geq h$ — ③

The result is consistent with the uncertainty relation $\Delta x \Delta p \geq \hbar/2$.

Arguments like this, although superficially attractive, must be approached with caution. The argument above implies that electron can possess a definite position and momentum at an instant and that it is the measurement process that introduces the indeterminancy in $\Delta x \Delta p$. On the contrary, this indeterminancy is inherent in the nature of a moving body.

uncertainty in energy and time —

Another form of the uncertainty principle concerns energy and time. we might wish to measure the energy E emitted during the time interval Δt in an atomic process. If the energy is in the form of em waves, the limited time available restricts the accuracy with which we can determine the frequency ν of the waves. Let us assume that the minimum uncertainty in the number of waves we count in a ~~[illegible]~~ wave group is one wave. Since the frequency of the waves under study is equal to the number of them we count divided by time interval, the uncertainty $\Delta\nu$ in our frequency measurement is $\Delta\nu \geq \frac{1}{\Delta t}$.

The corresponding energy uncertainty is, $\Delta E = h\Delta\nu$,

and so, $\Delta E \geq \frac{h}{\Delta t}$ or, $\Delta E \cdot \Delta t \geq h$

More precise calculation leads to $\Delta E . \Delta t \geq \hbar/2$

Examples 1. A measurement establishes the position of a proton with an accuracy of $\pm 1.00 \times 10^{-11}$ m. Find the uncertainty in the proton's position 1.00s later. Assume $v \ll c$.

ans :- $\Delta x \Delta p \geq \frac{\hbar}{2}$. If we call the uncertainty in the proton's position as Δx_0 at the time $t=0$, the uncertainty in momentum at this time is therefore -

$$\Delta p \geq \frac{\hbar}{2\Delta x_0}.$$

Since $v \ll c$, the momentum uncertainty is $\Delta p = \Delta(mv) = m\Delta v$ and the uncertainty in the proton's velocity is $\Delta v = \frac{\Delta p}{m} \geq \frac{\hbar}{2m\Delta x_0}$.

The distance x the proton covers in time t cannot be known more accurately than -

$$\Delta x = t\,\Delta v \geq \frac{\hbar t}{2m\Delta x_0}$$

Hence Δx is inversely proportional to Δx_0: the more we know about the proton's position at $t=0$, the less we know about its later position at $t>0$. The value of Δx at $t = 1.00$ s is -

$$\Delta x \geq \frac{(1.054 \times 10^{-34}\ J.s)(1.00s)}{2 \times (1.672 \times 10^{-27}\ kg)(1.00 \times 10^{-11}\ m)}$$

$$\geq \cancel{3.15 \times 10^{3}}\ m.$$

example 2 – A typical nucleus is about 5.0×10^{-15} m in radius. Use the uncertainty principle to place a lower limit on the energy an electron must have if it is to be part of the nucleus.

ans:- Letting $\Delta x = 5.0 \times 10^{-15}$ m, we have –

$$\Delta p \geq \frac{\hbar}{2\Delta x} \geq \frac{(1.054 \times 10^{-34} \text{ J.s})}{2 \times (5.0 \times 10^{-15} \text{ m})} \geq 1.1 \times 10^{-20} \text{ kg. m/s.}$$

If this is the uncertainty in a nuclear electron's momentum, the momentum p itself must be at least comparable in magnitude. An electron with such a momentum has a kinetic energy, KE many times greater than the rest mass energy mc^2, we can like $KE = pc$. Therefore, $KE \geq (1.1 \times 10^{-20} \text{ kg.m/s})(3.0 \times 10^8 \text{ m/s}) \geq 3.3 \times 10^{-12}$ J. Since $1\text{ eV} = 1.6 \times 10^{-19}$ J, the kinetic energy of an electron must exceed 20 MeV if it is to be inside a nucleus. Experiments show that the electrons emitted by certain unstable nuclei never have more than a small fraction of this energy, from which we conclude that nuclei cannot contain electrons. The electrons an unstable nuclei emit comes into being at the moment of the nucleus decay.

example 3 – A hydrogen atom is 5.3×10^{-11} m in radius. Use the uncertainty principle to estimate the minimum energy an ~~atom~~ electron can have in this atom.

ans:- $\Delta x = 5.3 \times 10^{-11}$ m, $\Delta p \geq \frac{\hbar}{2\Delta x} \geq 9.9 \times 10^{-25}$ kg. m/s.

An electron whose momentum is of this order of magnitude behaves as classical particle, and its kinetic energy is, $KE = \frac{p^2}{2m} \geq \frac{(9.9 \times 10^{-25})^2}{2(9.1 \times 10^{-31})} \geq 5.4 \times 10^{-19}$ J.

which is 3.4 eV. The KE of an electron in the lowest energy level of a hydrogen atom is actually 13.6 eV.

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example 4 – An excited atom give up its excess energy by emitting a photon of characteristic frequency. The average period that elapses between the excitation of an atom and the time it radiates is 1.0×10^{-8} s. Find the inherent uncertainty in the frequency of the photon.

ans:- The photon energy is uncertain by the amount –

$$\Delta E \geq \frac{\hbar}{2\Delta t} \geq \frac{(1.054 \times 10^{-34}\ J.s)}{2(1.0 \times 10^{-8}\ s)} \geq 5.3 \times 10^{-27}\ J.$$

The corresponding uncertainty in the frequency of light is –

$$\Delta \nu = \frac{\Delta E}{h} \gtrsim 8 \times 10^{6}\ Hz.$$

example 5 – obtain an expression for the energy levels (in MeV) of a neutron confined to a one dimensional box 1.00×10^{-14} m wide. What is the neutron's minimum energy?

ans:- $$E_n = \frac{n^2 h^2}{8mL^2} = n^2 \cdot \frac{(6.625 \times 10^{-34})^2}{8 \times (1.67 \times 10^{-27}) \times (1.00 \times 10^{-14})} = 3.28 \times 10^{-3} n^2\ J = 20.5\ n^2\ MeV$$

minimum energy corresponding to $n=1$ is, $E_1 = 20.5$ MeV.

example 6 A proton in a one-dimensional box has an energy of 400 keV, in its 1st excited state. How wide is the box?

ans:- 1st excited state $n=2$, so, $E_2 = \frac{4h^2}{8mL^2} \Rightarrow L = \sqrt{\frac{h^2}{8mE_2}} = 4.53 \times 10^{-14}$ m

example 7 The atoms in a solid possess a certain minimum zero point energy even at 0K, while no such restriction holds for the molecules in an ideal gas. Use the uncertainty principle to explain these statements.

ans:- Each atom in a solid is limited to a certain definite region in space – otherwise the assembly of atoms would not be solid. The uncertainty in position of each atom is therefore finite, and its momentum and hence energy cannot be zero. The position of an ideal-gas molecule is not restricted, so the uncertainty in its position is effectively infinite and its momentum and hence energy can be zero.

example 8 The position and momentum of a 1.00 keV electron are simultaneously determined. If its position is located within 0.100 nm, what is the percentage in uncertainty in its momentum?

ans:- $\Delta x . \Delta p \geq \frac{\hbar}{2}$, $\Delta p \geq \frac{\hbar}{2\Delta x}$, so uncertainty in momentum,

$$\Delta p \approx \frac{\hbar}{2\Delta x}, \quad so, \quad \frac{\Delta p}{p} = \frac{\hbar}{2p\Delta x} = \frac{h}{4\pi \Delta x . p} = \frac{h}{4\pi \Delta x . \sqrt{2mE}}$$

$$\therefore \frac{\Delta p}{p} = \frac{1.24 \times 10^{-6} \text{ eV.m}}{4 \times 3.142 \times (1.00 \times 10^{-10} \text{m}) \sqrt{2 \times (511 \times 10^{3} \text{eV}) \times (1.00 \times 10^{3} \text{eV})}} \quad \left(\because E = \frac{p^2}{2m}\right).$$

$$= 3.1 \times 10^{-2}$$
$$= 3.1\% .$$

example 9 - How accurately can the position of a proton with $v \ll c$ be determined without giving it more than 1.00 keV of kinetic energy?

ans:- The proton will need to move a minimum distance, $v\Delta t \geq v . \frac{\hbar}{2\Delta E}$

where $v = \sqrt{\frac{2(KE)}{m}} = \sqrt{\frac{2\Delta E}{m}}$, so that -

$$v\Delta t = \sqrt{\frac{2k}{m}} . \frac{h}{4\pi . \Delta E} = \frac{h}{2\pi\sqrt{2mk}} = \frac{hc}{2\pi\sqrt{2(mc^2)k}}, \quad k = \text{K.E. of proton.}$$

$$= \frac{1.24 \times 10^{-6} \text{ eV.m}}{2\pi\sqrt{2(938 \times 10^{6} \text{ MeV})(1.00 \times 10^{3} \text{ eV})}} = 0.144 \times 10^{-12} \text{m} = 0.144 \text{ pm}$$

example 10 — A marine radar operating at frequency of 9400 MHz emits groups of e.m. waves 0.800 μs in duration. The time needed for the reflections of these groups to return indicates the distance to a target. (a) Find the length of each group and the number of waves it contains. (b) What is the approximate minimum bandwidth (ie, frequency spread) the radar receiver must be able to process?

ans:- The length of each group is $c\Delta t = (3.0\times10^{8}\,m/s)(8.00\times10^{-5}s) = 24m$.
The no. of waves in each group is the pulse duration divided by the wave period, which is the pulse duration divided by the wave period, which is the pulse duration multiplied by the frequency, $(8.00\times10^{-8}s)(9400\times10^{6}\,Hz) = 752$ waves

The bandwidth is the reciprocal of the pulse duration —

$$(8.0\times10^{-8}s)^{-1} = 12.5\ MHz.$$

example 11 The frequency of oscillation of a harmonic oscillator of mass m and spring constant C is $\nu = \frac{\sqrt{C/m}}{2\pi}$. The energy of the oscillator is $E = \frac{p^2}{2m} + \frac{Cx^2}{2}$, where p is its momentum when its displacement from the equilibrium position is x. In classical physics the minimum energy of the oscillator $E_{min} = 0$. Use the uncertainty principle to find an expression for E in terms of x only and show that the minimum energy is actually $E_{min} = \frac{1}{2}h\nu$.

ans:- $\Delta p \sim p$ and $\Delta x \sim x$, so that $\Delta p . \Delta x \sim \frac{\hbar}{2}$ leads to, $p.x = \frac{\hbar}{2} = \frac{h}{4\pi}$

or, $p = \frac{h}{4\pi x}$ and $E = E(x) = \frac{h^2}{8\pi^2 m}\cdot\frac{1}{x^2} + \frac{C}{2}x^2$,

Now, setting $\frac{dE}{dx} = 0$, $-\frac{h^2}{4\pi^2 m}\cdot\frac{1}{x^3} + Cx = 0 \Rightarrow x^2 = \frac{\hbar}{\sqrt{mC}}$.

Substituting in $E(x)$, $E_{min} = \left(\frac{h^2}{8\pi^2 m}\right)\left(\frac{2\pi\sqrt{mC}}{h}\right) + \frac{C}{2}\cdot\frac{h}{2\pi\sqrt{mC}}$

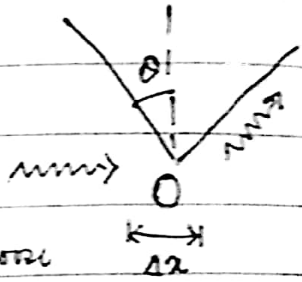
$\therefore E_{min} = \frac{h}{2\pi}\sqrt{\frac{C}{m}}$, Now, since, $\nu = \frac{1}{2\pi}\sqrt{\frac{C}{m}}$,

So, $E_{min} = \frac{h\nu}{2}$.

Heisenberg's γ-ray microscope -

A striking thought experiment illustrating uncertainty principle is the Heisenberg's γ-ray microscope. To observe a particle, say an electron, we shine it with light ray of wavelength λ and collect the Compton scattered light in a microscope objective whose diameter attends an angle θ with the electron.

The precision with which the electron can be located, Δx, is defined by resolving power of the microscope,



$$\sin\theta = \frac{\lambda}{\Delta x} \Rightarrow \Delta x = \frac{\lambda}{\sin\theta} \quad \text{—(1)}$$

It appears that by making λ small (that is why we chose γ-ray) and by making $\sin\theta$ large, Δx can be made as small as desired. But, according to uncertainty principle, we can do so only at the expense of our knowledge of x-component of electron momentum. In order to record the Compton scattered photon by the microscope, the photon must stay in the cone of angle θ and hence its x-component of the momentum can vary within $\pm \frac{h}{\lambda}\sin\theta$. This implies the magnitude of the recoil momentum of the electron is uncertain by, $\Delta p_x = \frac{2h}{\lambda}\sin\theta$ —(2)

The product of the uncertainty yields $\Delta x \Delta p_x = \frac{\lambda}{\sin\theta} \cdot \frac{2h}{\lambda}\sin\theta = 4\pi\hbar$