Multivariate Normal Distribution

The p.d.f. of univariate normal distribution can be written as

$$f(x) = ke^{-\frac{1}{2}\alpha(x-\beta)^2} \qquad x \in \mathbb{R}$$

where $\alpha > 0$ and k is obtained such that $\int\limits_{-\infty}^{\infty} f(x)dx = 1$.

Now, suppose $X = (X_1, X_2, \dots, X_p)'$ be a p dimensional random vector. The multivariate normal distribution of X has an analogous form where scaler x is replaced by a vector $x = (x_1, x_2, \dots, x_p)'$, the scaler constant β is replaced a vector $b = (b_1, b_2, \dots, b_p)'$ and positive constant α is replaced by positive definite real symmetric matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1p} \\ a_{21} & a_{22} & \dots & a_{2p} \\ \vdots & \vdots & \dots & \vdots \\ a_{p1} & a_{p2} & \dots & a_{pp} \end{pmatrix}$$

The square $\alpha(x-\beta)^2 = (x-\beta)\alpha(x-\beta)$ is replaced by the quadratic form

$$(x-b)'A(x-b) = \sum_{i,j=1}^{p} a_{ij}(x_i - b_i)(x_j - b_j).$$

So, the density function of p variate normal distribution is

$$f(x_1, x_2, \dots, x_p) = Ke^{-\frac{1}{2}(x-b)'A(x-b)}$$

where K is chosen such that

$$\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\cdots\int\limits_{-\infty}^{\infty} f(x_1, x_2, \dots, x_p)dx_1dx_2\dots dx_p = 1$$

As $(x-b)'A(x-b)$ is positive definite, we have $f(x_1, x_2, \ldots, x_p) \leq K$ that is f is bounded for all $x \in \mathbb{R}^p$.

If A is positive definite there exists a non singular matrix C such that

$$C'AC = I.$$

Let us consider the transformation $(x_1, x_2, \ldots, x_p)' \to (y_1, y_2, \ldots, y_p)'$ such that

$$x - b = Cy.$$

$$(x-b)'A(x-b) = y'C'ACy = y'y.$$

The jacobian of the transformation is $J = \mod |C|$ = absolute value of the determinant C.

Thus

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} Ke^{-\frac{1}{2}(x-b)'A(x-b)}dx_1dx_2\ldots dx_p &= 1\\
\Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} Ke^{-\frac{1}{2}y'y} \mod |C|dy_1dy_2\ldots dy_p &= 1\\
\Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} Ke^{-\frac{1}{2}\sum_{i=1}^{p} y_i^2} \mod |C|dy_1dy_2\ldots dy_p &= 1\\
\Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} K \mod |C|\prod_{i=1}^{p} e^{-\frac{1}{2}y_i^2}dy_1dy_2\ldots dy_p &= 1\\
\Rightarrow K \mod |C|\int_{-\infty}^{\infty} e^{-\frac{1}{2}y_1^2}dy_1 \int_{-\infty}^{\infty} e^{-\frac{1}{2}y_2^2}dy_2\ldots\int_{-\infty}^{\infty} e^{-\frac{1}{2}y_p^2}dy_p &= 1\\
\Rightarrow K. \mod |C|\left(\sqrt{2\pi}\right)^p &= 1
\end{aligned}$$

Now, $|C'| \cdot |A| \cdot |C| = 1$, hence $|C| = 1/\sqrt{|A|}$, so

$$K = \frac{\sqrt{|A|}}{(2\pi)^{p/2}}$$

So, the probability density function of multivariate normal distribution is

$$\frac{\sqrt{|A|}}{(2\pi)^{p/2}} e^{-\frac{1}{2}(x-b)'A(x-b)}$$

Now we want to write b and $|A|$ in terms of moments. We define

$$\mu = E(X) = (E(X_1), E(X_2), \ldots, E(X_p))' = (\mu_1, \mu_2, \ldots, \mu_p)'$$

is the mean vector of X and

$$D(X) = \Sigma = E(X-\mu)(X-\mu)' = E(XX') - \mu\mu'$$

is variance - covariance matrix.

The transformation gives $X = CY + b$. Now $Y_1, Y_2, \ldots, Y_p$ are i.i.d. normal random variable with $E(Y_i) = 0$ and $V(Y_i) = 1$ for all $i = 1, 2, \ldots, p$. That gives $E(Y) = (0, 0, \ldots, 0)'$ and hence $\mu = E(X) = b$. Again the variance covraiance matrix

$$E(YY') = I_p.$$

Thus

$$E(X-\mu)(X-\mu)' = CIC' = CC'.$$

Now, as $C'AC = I$ we get $A = (C')^{-1}C^{-1}$ by multiplication of $(C')^{-1}$ on the left and C^{-1} on the right. So,

$$CC' = A^{-1}$$

So, the variance-covariance matrix of X is

$$\Sigma = E(X-\mu)(X-\mu)' = A^{-1}.$$

Here Σ is positive definite.

We can summarize as follows, given a vector $\mu = (\mu_1, \mu_2, \ldots, \mu_p)'$ and positive definite matrix Σ there exists random vector $X = (X_1, X_2, \ldots, X_p)'$ that follows multivariate normal distribution with density function

$$f(x_1, x_2, \ldots, x_p) = \frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}$$

with $x = (x_1, x_2, \ldots, x_p)' \in \mathbb{R}^p$.

Exercises:

1. Derive the p.d.f. of bivariate normal random vector $X = (X_1, X_2)'$ with mean vector $(\mu_1, \mu_2)'$ and variance-covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

2. Let us consider the following densities

   (a) $\frac{1}{2\pi}\exp[-\frac{1}{2}(x^2 + y^2 + 4x - 6y + 13)]$

   (b) $\frac{1}{2\pi}\exp[-\frac{1}{2}(2x^2 + y^2 + 2xy - 22x - 14y + 65)]$

   obtain b and A. Also find C such that $CAC' = I$

Theorem 1: Let $\mathbf{X}$ be p component random vector with $\mathbf{X} \sim N_p(\mu, \mathbf{\Sigma})$. Then

$$\mathbf{Y} = C\mathbf{X}$$

is distributed according to $N_p(C\mu, C\mathbf{\Sigma}C')$ for C non singular.

Proof: The transformation $\mathbf{Y} = C\mathbf{X}$ gives

$$x = C^{-1}y$$

So, the jacobian of the transformation is

$$\text{mod } |C^{-1}| = \frac{1}{\text{mod } |C|} = \sqrt{\frac{1}{|C||C'|}} = \sqrt{\frac{|\mathbf{\Sigma}|}{|C|\cdot|\mathbf{\Sigma}|\cdot|C'|}} = \sqrt{\frac{|\mathbf{\Sigma}|}{|C\mathbf{\Sigma}C'|}}$$

The quadrattic component of $N_p(\mu, \mathbf{\Sigma})$ is

$$Q = (x-\mu)'\Sigma^{-1}(x-\mu)$$

The transformation $x = C^{-1}y$ gives

$$\begin{aligned} Q &= (C^{-1}y-\mu)'\Sigma^{-1}(C^{-1}y-\mu) \\ &= (C^{-1}y - CC^{-1}\mu)'\Sigma^{-1}(C^{-1}y - CC^{-1}\mu) \\ &= \left[(C^{-1})(y-C\mu)\right]'\Sigma^{-1}\left[C^{-1}(y-C\mu)\right] \\ &= (y-C\mu)'(C^{-1})'\Sigma^{-1}C^{-1}(y-C\mu) \\ &= (y-C\mu)'(C\Sigma C')^{-1}(y-C\mu) \end{aligned}$$

since $(C')^{-1} = (C^{-1})'$.

Thus the density of $\mathbf{Y}$ is

$$\frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}}e^{-\frac{1}{2}(C^{-1}y-\mu)'\Sigma^{-1}(C^{-1}y-\mu)} \quad \text{mod } |C^{-1}| = \frac{1}{(2\pi)^{\frac{p}{2}}|C\mathbf{\Sigma}C'|^{\frac{1}{2}}}e^{-\frac{1}{2}(y-C\mu)'(C\Sigma C')^{-1}(y-C\mu)}$$

Exercise: 1. Let $\mathbf{Y} = C\mathbf{X} + b$ where C is $p \times p$ non singular matrix and b is $p \times 1$ vector. Obtain the distribution of $\mathbf{Y}$ if $\mathbf{X} \sim N_p(\mu, \mathbf{\Sigma})$. Show that $\mathbf{Y} \sim N_p(C\mu + b, C\mathbf{\Sigma}C')$.

2. Show that if $\mathbf{\Sigma}$ is positive definite then $\mathbf{\Sigma}^{-1}$ is also positive definite.

Marginal and Conditional Distribution

Let $\mathbf{X}^{p\times 1}$ be random vector which follows a p dimensional multivariate normal $N_p(\mu, \mathbf{\Sigma})$. Let $\mathbf{X}$ be partitioned as

$$\mathbf{X} = \begin{pmatrix} X^{(1)} \\ X^{(2)} \end{pmatrix} \qquad \text{where} \quad X^{(1)} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_q \end{pmatrix} \quad \text{and} \quad X^{(2)} = \begin{pmatrix} X_{q+1} \\ X_{q+2} \\ \vdots \\ X_p \end{pmatrix}$$

We assume $\mathbf{X}$ is p-variate normal with mean vector

$$\mu = \begin{pmatrix} \mu^{(1)} \\ \mu^{(2)} \end{pmatrix}$$

and variance covariance matrix

$$\mathbf{\Sigma} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

where $\Sigma_{21} = \Sigma'_{12}$. Here

$$E(X^{(1)}) = \mu^{(1)} \quad \text{and} \quad E(X^{(2)}) = \mu^{(2)}$$

and

$$E\left(X^{(1)} - \mu^{(1)}\right)\left(X^{(1)} - \mu^{(1)}\right)' = \Sigma_{11}$$
$$E\left(X^{(1)} - \mu^{(1)}\right)\left(X^{(2)} - \mu^{(2)}\right)' = \Sigma_{12}$$
$$E\left(X^{(2)} - \mu^{(2)}\right)\left(X^{(2)} - \mu^{(2)}\right)' = \Sigma_{22}$$

Theorem 2 $X^{(1)}$ and $X^{(2)}$ are independently if $\Sigma_{12} = \Sigma'_{21} = \mathbf{0}$.

Under the given condition

$$\mathbf{\Sigma} = \begin{pmatrix} \Sigma_{11} & \mathbf{0} \\ \mathbf{0} & \Sigma_{22} \end{pmatrix}.$$

Its inverse is

$$\mathbf{\Sigma}^{-1} = \begin{pmatrix} \Sigma_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \Sigma_{22}^{-1} \end{pmatrix}.$$

The quadratic form is

$$\begin{aligned}
Q &= (x-\mu)'\Sigma^{-1}(x-\mu) \\
&= \left[(x^{(1)}-\mu^{(1)})', (x^{(2)}-\mu^{(2)})'\right] \begin{pmatrix} \Sigma_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \Sigma_{22}^{-1} \end{pmatrix} \begin{pmatrix} x^{(1)}-\mu^{(1)} \\ x^{(2)}-\mu^{(2)} \end{pmatrix} \\
&= \left(x^{(1)}-\mu^{(1)}\right)' \Sigma_{11}^{-1} \left(x^{(1)}-\mu^{(1)}\right) + \left(x^{(2)}-\mu^{(2)}\right)' \Sigma_{22}^{-1} \left(x^{(2)}-\mu^{(2)}\right) \\
&= Q_1 + Q_2
\end{aligned}$$

where $Q_1 = \left(x^{(1)}-\mu^{(1)}\right)' \Sigma_{11}^{-1} \left(x^{(1)}-\mu^{(1)}\right)$ and $Q_2 = \left(x^{(2)}-\mu^{(2)}\right)' \Sigma_{22}^{-1} \left(x^{(2)}-\mu^{(2)}\right)$.

Also we note that $|\Sigma| = |\Sigma_{11}| \cdot |\Sigma_{22}|$.

The pdf of $\mathbf{X}$ is

$$\begin{aligned}
f(x|\mu,\Sigma) &= \frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}Q} \\
&= \frac{1}{(2\pi)^{\frac{q}{2}}|\Sigma_{11}|^{\frac{1}{2}}} e^{-\frac{1}{2}Q_1} \cdot \frac{1}{(2\pi)^{\frac{p-q}{2}}|\Sigma_{22}|^{\frac{1}{2}}} e^{-\frac{1}{2}Q_2} \\
&= f_1(x^{(1)}|\mu^{(1)}, \Sigma_{11}) \cdot f_2(x^{(2)}|\mu^{(2)}, \Sigma_{22})
\end{aligned}$$

So, marginal distribution of $X^{(1)}$ is $N_q(\mu^{(1)}, \Sigma_{11})$ and marginal distribution of $X^{(2)}$ is $N_{p-q}(\mu^{(2)}, \Sigma_{22})$.

Let us consider the linear transformation

$$Y = \begin{pmatrix} Y^{(1)} \\ Y^{(2)} \end{pmatrix} = \begin{pmatrix} I_q & -B \\ 0 & I_{p-q} \end{pmatrix} \begin{pmatrix} X^{(1)} \\ X^{(2)} \end{pmatrix} = CX$$

If $X \sim N_p(\mu, \Sigma)$, then by Theorem 1, $Y \sim N_p(C\mu, C\Sigma C')$ where

$$C\mu = \begin{pmatrix} \mu^{(1)} - B\mu^{(2)} \\ \mu^{(2)} \end{pmatrix}$$

and

$$C\Sigma C' = \begin{pmatrix} \Sigma_{11} + B\Sigma_{22}B' - B\Sigma_{21} - \Sigma_{12}B' & \Sigma_{12} - B\Sigma_{22} \\ \Sigma_{12} - B\Sigma_{22} & \Sigma_{22} \end{pmatrix}$$

As Σ is non -singular both Σ_{11}^{-1} and Σ_{22}^{-1} exists. If B is chosen such that $\Sigma_{12} - B\Sigma_{22} = 0$ that is if

$$B = \Sigma_{12}\Sigma_{22}^{-1}$$

then $Y^{(1)}$ and $Y^{(2)}$ are uncorrelated and thus independent.

We have

$$\begin{pmatrix} Y^{(1)} \\ Y^{(2)} \end{pmatrix} = \begin{pmatrix} X^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}X^{(2)} \\ X^{(2)} \end{pmatrix} \sim N_p\left(\begin{pmatrix} \nu_{1.2} \\ \mu^{(2)} \end{pmatrix}, \begin{pmatrix} \Sigma_{11.2} & \mathbf{0} \\ \mathbf{0} & \Sigma_{22} \end{pmatrix}\right)$$

where

$$\nu_{1.2} = \mu^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\mu^{(2)}$$

and

$$\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

Since, $X^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}X^{(2)}$ and $X^{(2)}$ are independent, the marginal densities are

$$g\left(x^{(1)}|x^{(2)}\right) = \frac{1}{(2\pi)^{\frac{q}{2}}|\Sigma_{11.2}|^{\frac{1}{2}}} e^{-\frac{1}{2}(x^{(1)}-\nu_{1.2})'\Sigma_{11.2}^{-1}(x^{(1)}-\nu_{1.2})}$$

and

$$f_2(x^{(2)}) = \frac{1}{(2\pi)^{\frac{p-q}{2}}|\Sigma_{22}|^{\frac{1}{2}}} e^{-\frac{1}{2}\left(x^{(2)}-\mu^{(2)}\right)'\Sigma_{22}^{-1}\left(x^{(2)}-\mu^{(2)}\right)}$$

So, using linear transformation we can rewrite the joint density of $X = (X^{(1)}, X^{(2)})'$ as

$$f(x|\mu, \Sigma) = g\left(x^{(1)}|x^{(2)}\right) f_2(x^{(2)})$$

But

$$f(x^{(1)}, x^{(2)}|\mu, \Sigma) = f_{1|2}\left(x^{(1)}|x^{(2)}\right) f_2(x^{(2)})$$

where $f_{1|2}\left(x^{(1)}|x^{(2)}\right)$ is the conditional density function of $X^{(1)}$ given $X^{(2)}$. So, conditional density of $X^{(1)}$ given $X^{(2)} = X^{(2)}$ must be $g\left(x^{(1)}|x^{(2)}\right)$. Since the quadratic form of $g\left(x^{(1)}|x^{(2)}\right)$ can be written as

$$Q(x^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}x^{(2)}; \nu_{1.2}, \Sigma_{11.2}) = \left(x^{(1)} - \mu_{1.2}\right)' \mathbf{\Sigma}_{11.2} \left(x^{(1)} - \mu_{1.2}\right)$$

where

$$\mu_{1.2} = \mu^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(x^{(2)} - \mu^{(2)})$$

and

$$\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}.$$

So,

$$X^{(1)}|X^{(2)} \sim N_q\left(\mu^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(x^{(2)} - \mu^{(2)}), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right)$$

Exercise Obtain the conditional density of $X^{(2)}$ given $X^{(1)}$

Moment Generating function

The moment generating function of $\mathbf{X}^{p\times 1}$ distributed according as $N_p(\mu, \mathbf{\Sigma})$ is

$$M_{\mathbf{X}}(t) = \exp\left\{t'\mu + \frac{1}{2}t'\Sigma t\right\}$$

We have

$$M_{\mathbf{X}}(t) = \frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)+t'x}dx_1dx_2\ldots dx_p$$

Let $y = x - \mu$, we obtain

$$M_X(t) = \frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}}e^{t'\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{1}{2}y'\Sigma^{-1}y+t'y}dy$$

Since Σ is positive definite, so Σ^{-1} is also positive definite. Hence $\Sigma^{-1} = H'H$ for some non singular matrix H. Then $|H|^2 = \frac{1}{|\Sigma|}$. Let us consider the tranformation $z = Hy$. The jacobian of the transformation $\frac{1}{|H|} = |\Sigma|^{\frac{1}{2}}$. So, $dy = |\Sigma|^{\frac{1}{2}}dz$. So,

$$\begin{aligned}
M_{\mathbf{X}}(t) &= \frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}}e^{t'\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{1}{2}y'H'Hy+t'y}dy \\
&= \frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}}e^{t'\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{1}{2}z'z+t'H^{-1}z}|\Sigma|^{\frac{1}{2}}dz \\
&= \frac{1}{(2\pi)^{\frac{p}{2}}}e^{t'\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sum\limits_{i=1}^{p}\left(z_i^2-2b_iz_i\right)}dz
\end{aligned}$$

where $b' = (b_1, b_2, \ldots, b_p) = t'H^{-1}$. So,

$$
\begin{aligned}
M_{\mathbf{X}}(t) &= \frac{1}{(2\pi)^{\frac{p}{2}}} e^{t'\mu + \frac{1}{2}b'b} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sum_{i=1}^{p}(z_i - b_i)^2} dz \\
&= \exp\left\{t'\mu + \frac{1}{2}b'b\right\} \\
&= \exp\left\{t'\mu + \frac{1}{2}t'H^{-1}(H')^{-1}t\right\} \\
&= \exp\left\{t'\mu + \frac{1}{2}t'(H'H)^{-1}t\right\} \\
&= \exp\left\{t'\mu + \frac{1}{2}t'\Sigma t\right\}
\end{aligned}
$$

Theorem If $X \sim N_p(\mu, \mathbf{\Sigma})$ then the linear combination $l'X$ follows univariate normal given by $l'X \sim N(l'\mu, l'\Sigma l)$ where $l' = (l_1, l_2, \ldots, l_p)$.

Proof: Left as an exercise. Use moment generating function.

Theorem: If $\mathbf{X} \sim N_p(\mu, \mathbf{\Sigma})$ and $\mathbf{Y} = \mathbf{CX} + \mathbf{b}$ where C is any given $q \times p$ real matrix with $Rank(C) = q \leq p$ and $\mathbf{b}$ is any $q \times 1$ vector , then $\mathbf{Y} \sim N_q(\mathbf{C}\mu + \mathbf{b}, \mathbf{C\Sigma C'})$.

Proof: Let us consider the transformation

$$\mathbf{Y}^* = \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{C} \\ \mathbf{B} \end{pmatrix} \mathbf{X} + \begin{pmatrix} \mathbf{b} \\ \mathbf{0}_{p-q} \end{pmatrix}$$

where $\mathbf{B}$ is any $(p-q) \times p$ matrix. Then

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} \sim N_p\left(\begin{pmatrix} \mathbf{C}\mu + \mathbf{b} \\ \mathbf{B} \end{pmatrix}, \begin{pmatrix} \mathbf{C\Sigma C'} & \mathbf{C\Sigma B'} \\ \mathbf{B\Sigma C'} & \mathbf{B\Sigma B'} \end{pmatrix}\right)$$

Then $\mathbf{Y} = \mathbf{Y}_1 = \mathbf{CX} + \mathbf{b} \sim N_q\left(\mathbf{C}\mu + \mathbf{b}, \mathbf{C\Sigma C'}\right)$.

Theorem: If $\mathbf{X} \sim N_p(\mu, \mathbf{\Sigma})$, then

$$(\mathbf{X} - \mu)'\Sigma^{-1}(\mathbf{X} - \mu) \sim \chi^2_p$$

Proof: As Σ^{-1} is positive definite we have $\Sigma^{-1} = H'H$ for some non singular matrix H. Let $\mathbf{Z} = H(\mathbf{X} - \mu)$. Then jacobian of the transformation is $|\mathbf{J}| = \frac{1}{|H|} = |\mathbf{\Sigma}|^{\frac{1}{2}}$. The joint pdf of $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_p)'$ is

$$f(z) = \frac{1}{(2\pi)^{\frac{p}{2}}} \exp\left\{-\frac{1}{2}z'z\right\} = \frac{1}{(2\pi)^{\frac{p}{2}}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{p} z_i^2\right\}$$

So, $\mathbf{Z}_i \sim N(0,1)$ for all $i = 1, 2, \ldots, p$. Hence

$$(\mathbf{X} - \mu)'\Sigma^{-1}(\mathbf{X} - \mu) = \mathbf{Z}'\mathbf{Z} \sim \chi_p^2$$

Regression and Correlation

Let $\mathbf{X}^{p\times 1}$ be random vector which follows a p dimensional multivariate normal $N_p(\mu, \mathbf{\Sigma})$. Let $\mathbf{X}$ be partitioned as

$$\mathbf{X} = \begin{pmatrix} X^{(1)} \\ X^{(2)} \end{pmatrix} \qquad \text{where} \quad X^{(1)} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_q \end{pmatrix} \quad \text{and} \quad X^{(2)} = \begin{pmatrix} X_{q+1} \\ X_{q+2} \\ \vdots \\ X_p \end{pmatrix}$$

We assume $\mathbf{X}$ is p-variate normal with mean vector

$$\mu = \begin{pmatrix} \mu^{(1)} \\ \mu^{(2)} \end{pmatrix}$$

and variance covariance matrix

$$\mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{pmatrix}$$

where $\mathbf{\Sigma}_{21} = \mathbf{\Sigma}_{12}'$.

Then conditional distribution of $\mathbf{X}^{(1)}$ given $\mathbf{X}^{(2)}$ is given by ,

$$\mathbf{X}^{(1)}|\mathbf{X}^{(2)} \sim N_q\left(\mu_{1.2}, \mathbf{\Sigma}_{11.2}\right)$$

where $\mu_{1.2} = \mu^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(x^{(2)} - \mu^{(2)})$ and $\mathbf{\Sigma}_{11.2} = \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}$.

Then $\mathbf{E}\left(\mathbf{X}^{(1)}|\mathbf{X}^{(2)}\right) = \mu^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(x^{(2)} - \mu^{(2)})$.

Definition: The matrix $\mathbf{B} = \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}$ is the matrix of regression coeffcients of $\mathbf{X}^{(1)}$ on $\mathbf{x}^{(2)}$.

The element in the ith row and $(k-q)$th column of $\mathbf{B} = \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}$ is denoted by

$$\beta_{ik.q+1\ldots k-1,k+1,\ldots p}, \qquad i = 1, 2, \ldots, q, k = q+1, \ldots, p$$

The vector $\mu^{(1)} + \mathbf{B}\left(\mathbf{x}^{(2)} - \mu^{(2)}\right)$ is called regression function.

Let $\sigma_{ij.q+1\ldots p}$ is the i, jth element of $\mathbf{\Sigma}_{11.2}$.

Definition:

$$\rho_{ij.q+1\ldots p} = \frac{\sigma_{ij.q+1\ldots p}}{\sqrt{\sigma_{ii.q+1\ldots p}}\sqrt{\sigma_{jj.q+1\ldots p}}} \qquad i, j = 1, 2, \ldots, q$$

is the partial correlation *between* X_i *and* X_j *holding* $X_{q+1}, \ldots, X_p$ *fixed.*

Let $\sigma'_{(i)}$ is the ith row of Σ_{12} and $\beta'_{(i)}$ is the ith row of $\mathbf{B}$ that is $\beta'_{(i)} = \sigma'_{(i)}\Sigma_{22}^{-1}$.

Definition

The correlation between X_i and $\beta'_{(i)}\mathbf{X}^{(2)}$ is called multiple correlation between X_i and $\mathbf{X}^{(2)}$.

It can be shown that multiple correlation between X_i and $\mathbf{X}^{(2)}$ is

$$\bar{R}_{i.q+1\ldots p} = \left(\frac{\sigma'_{(i)}\Sigma_{22}^{-1}\sigma_{(i)}}{\sigma_{ii}}\right)^{\frac{1}{2}}$$

Tests of Hypotheses for partial correlation coefficients

Let $r_{ij.q+1...p}$ is the sample partial correlation coefficient based on a sample of size N from a p dimensional multivariate normal distribution with population partial correlation coefficient $\rho_{ij.q+1...p}$.

To test $H_0 : \rho_{ij.q+1...p} = \rho_0$ against two test alternatives we can use Fisher's z for an approximate test based on large sample size N. Let

$$z = \frac{1}{2} \ln \frac{1 + r_{ij.q+1...p}}{1 - r_{ij.q+1...p}}$$

and

$$\zeta_0 = \frac{1}{2} \ln \frac{1 + \rho_0}{1 - \rho_0}.$$

Then H_0 is rejected if

$$|\sqrt{N - (p - q) - 3}(z - \zeta_0)| > \tau_{\alpha/2}$$

where τ_α is $100\alpha\%$ percentile point of standard normal distribution.

Test of Hypotheses for multiple correlation coeffcient

Let R be the sample correlation coefficient between $\mathbf{X_1}$ and $\mathbf{X}^{(2)} = (X_2, X_3, \ldots, X_p)$ based on a sample of size N from $N(\mu, \Sigma)$. If the population multiple correlation coefficient $\bar{R} = 0$ then $\left[\frac{R^2}{1-R^2}\right]\left[\frac{N-p}{p-1}\right]$ is distributed as F with $p - 1$ and $N - p$ degrees of freedom.

To test $H_0 : \bar{R} = 0$, we use the critical region

$$\frac{R^2}{1 - R^2} \frac{N - p}{p - 1} > F_{\alpha,p-1,N-p}$$

where $F_{\alpha,p-1,N-p}$ is the upper α significance point corresponding to α significance level.