Collective Risk Models for Short-Term Periods

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In *collective risk model* we assume a random process that generates claims for a portfolio of policies. This process is characterized in terms of portfolio as a whole rather than individual policies comprising the portfolio. In individual risk model, we had assumed that the number of times the claim will be made within the insured period is 0 or 1 and number of policies is fixed as n. But at the begining of a period of insurance cover, the company does not know that how many claims will occur and what the amounts of the claims will be. The *collective risk models* takes into account these two sources of variability.

Suppose N be the random variable that denotes the number of claims produced by a portfolio of policies in a given period. Then the possible values of N are $0, 1, 2, \ldots$. If X_i denotes the amount of the i^{th} claim in the given period, then the aggregate of claims S generated by the portfolio for a given period is modelled as

$$S = \sum_{i=1}^{N} X_i \quad \text{with} \quad S = 0 \quad \text{when} \quad N = 0.$$

Assumptions: (i) $X_1, X_2, \ldots$ are independent and identically distributed random variables .
(ii)N is independent of $(X_i)_{i=1}^{\infty}$.

Let $P(x)$ denote the common d.f. of the independent and identically distributed $X_i's$. Let X be a random variable with this d.f. Let

$$p_k = E(X^k)$$

, then expectation of S is

$$E(S) = E(E(S|N)) = E(N \cdot p_1) = p_1 E(N)$$

and variance is S, and

$$V(S) = V(E(S|N)) + E(V(S|N)) = V(p_1 N) + E(N(p_2 - p_1^2)) = p_1^2 V(N) + (p_2 - p_1^2) E(N)$$

So, expected aggregate claim is product of expected number of claims and expected individual claim amount and variance of S has two parts, first due to variance of number of claims and second due variance of individual claim amount.

Let $M_X(t) = E(e^{tX})$ is MGF of X and $M_N(t) = E(e^{tN})$ is MGF of N, then moment generating function of S is

$$\begin{aligned} M_S(t) &= E(e^{tS}) \\ &= E\left[E\left(e^{t\sum_{i=1}^{N} X_i} |N\right)\right] \\ &= E[(M_X(t))^N |N] \\ &= E[\exp(N \ln M_X(t)|N)] \\ &= M_N[\ln M_X(t)] \end{aligned}$$

<u>Example 1:</u>Let N is a geometric random variable with p.m.f. as

$$p(x) = pq^n \quad n = 0, 1, 2, \ldots$$

where $0 < q < 1$. Obtain the moment generating function of $S = \sum_{i=1}^{N} X_i$ in terms of MGF of X, $M_X(t)$ when N is assumed to be independent of i.i.d. random variables $X_1, X_2, \ldots$.

Now, the MGF of N is

$$M_N(t) = \sum_{n=0}^{\infty} e^{tn} pq^n = \frac{p}{1 - qe^t}.$$

As, $S = \sum_{i=1}^{N} X_i$ where N is assumed to be independent of i.i.d. random variables $X_1, X_2, \ldots$, the MGF of S can be written as

$$M_S(t) = M_N[\ln M_X(t)] = \frac{p}{1 - qe^{\ln M_X(t)}} = \frac{p}{1 - qM_X(t)}$$

The distribution of S

Let $G(x) = P(S \leq x)$ be distribution function of X and $F(x) = P(X_i \leq x)$ is the common distribution function of $X_i, i = 1, 2, \ldots$. Let $p_n = P(N = n)$ is the probability mass function of N where $n = 0, 1, 2, \ldots$. Now,

$$\{S \leq x\} = \bigcup_{n=1}^{\infty} \{S \leq x, N = n\}$$

So, for $x \geq 0$,

$$\begin{aligned} P(S \leq x) &= \sum_{n=0}^{\infty} P(S \leq x, N = n) \\ &= \sum_{n=0}^{\infty} P\left(\sum_{i=1}^{n} X_i \leq x | N = n\right) P(N = n) \\ &= \sum_{n=0}^{\infty} P\left(\sum_{i=1}^{n} X_i \leq x\right) P(N = n) \\ &= F^{n*}(x) p_n \end{aligned}$$

where F^{n*} is the distribution function of $X_1 + X_2 + \ldots + X_n$ and $F^{0*}(x) = 1$ if $x \geq 0$ and $= 0$ if $x < 0$.

Example 2: As in Example 1, let the numner of claims N is a geometric random variable with

$$p(x) = pq^n \quad n = 0, 1, 2, \ldots$$

and commnon ditribution unction of X is

$$F(x) = 1 - e^{-x}, \quad x > 0$$

that is individual claim amount is exponential random variable with mean 1. Then show that

$$M_S(t) = p + q\frac{p}{p - t}.$$

Solution: The MGF of X is

$$M_X(t) = \int_0^\infty e^{tx} e^{-x} dx = (1-t)^{-1}.$$

Then from the result in Example 1,

$$\begin{aligned} M_S(t) = M_N[\ln M_X(t)] &= \frac{p}{1 - qM_X(t)} \\ &= \frac{p}{1 - q(1-t)^{-1}} \\ &= \frac{p(1-t)}{p-t} \\ &= p + (1-p)\frac{p}{p-t} \end{aligned}$$

As 1 is the MGF of the random variable which is degenerated at 0 and $p/(p-t)$ is the MGF of exponential random variable with cdf $H(x) = 1 - e^{-px}$. Hence we S has a mixed distribution with weight p at 0 and weight q for $x > 0$. The cdf of S is

$$F_S(x) = p \cdot 1 + q \cdot (1 - e^{-px}) = 1 - qe^{-px}, \quad x \geq 0$$

Example 3 An insurance portfolio produces N claims where N is random variable defined as

$$N = \begin{cases} 0 & \text{with probability} \quad 0.5 \\ 1 & \text{with probability} \quad 0.4 \\ 3 & \text{with probability} \quad 0.1 \end{cases}.$$

The individual claim X(in lakhs) follows the probability distribution give as

$$X = \begin{cases} 1 & \text{with probability} \quad 0.9 \\ 10 & \text{with probability} \quad 0.1 \end{cases}.$$

It is assumed that the individual claim amount and N are independent random variables. Calculate the exact probability that aggregate claims exceed 3 times the expected claims.

Solution $E(N) = 0.7$ and $E(X) = 1.9$. Hence $E(S) = E(N)E(X) = 1.33$. Hence, required probability is

$$\begin{aligned} P(S > 3(1.33)) = P(S > 3.99) &= 1 - P(S < 3.99) \\ &= 1 - P(S = 0|N = 0)P(N = 0) \\ &\quad - P(X_1 = 1|N = 1)P(N = 1) \\ &\quad - P(X_1 = 1, X_2 = 1, X_3 = 1|N = 3)P(N = 3) \\ &= 1 - 1(0.5) - 0.9(0.4) - (0.9)^3 \cdot (0.1) \\ &= 0.0671 \end{aligned}$$

Distribution of N

1. Let the distribution of N is Poisson with parameter λ. Then $E(N) = Var(N) = \lambda$. With this choice of distribution of N, the distribution of S is called ***compound Poisson distribution***. As before, we assume $p_k = E(X_i^k)$, then expectation is

$$E(S) = \lambda p_1$$

and variance is

$$V(S) = V(E(S|N)) + E(V(S|N)) = p_1^2\lambda + (p_2 - p_1^2)\lambda = \lambda p_2.$$

The moment generating function of N is

$$M_N(t) = \exp\{\lambda(e^t - 1)\}.$$

Hence, moment generating function of S is

$$M_S(t) = M_N[\ln M_X(t)] = \exp\{\lambda(M_X(t) - 1)\}$$

2. When the variance number of claims exceeds its mean, then Poisson distribution is not appropriate. Suppose the distribution of N is negative binomial with p.m.f.

$$P(N = n) = \binom{r+n-1}{n} p^r q^n \qquad n = 0, 1, 2, \ldots.$$

This distribution has two parameters r and $0 < p < 1$. For this distribution, we have

$$M_N(t) = \left(\frac{p}{1-qe^t}\right)^r$$

and $E(X) = \frac{rq}{p}$ and $V(X) = \frac{rq}{p^2}$. When negative binomial is chosen as the distribution of N, then distribution of S is called *compound negative binomial* distribution. As, $S = \sum_{i=1}^{N} X_i$ and $E(X_i^k) = p_k$ we have

$$E(S) = \frac{rq}{p} p_1$$

and variance is

$$V(S) = V(E(S|N)) + E(V(S|N)) = p_1^2 \frac{rq}{p} + (p_2 - p_1^2)\frac{rq}{p^2} = p_2 \frac{rq}{p} + p_1^2 \frac{rq^2}{p^2}.$$

Also, the MGF of S is

$$M_S(t) = M_N[\ln M_X(t)] = \left(\frac{p}{1-qM_X(t)}\right)^r$$

<u>*Exercises:*</u>

1. Suppose that S has a compound Poisson distribution with $\lambda = 3$ and the common distribution of $X_1, X_2, \ldots$ is charaterrized by the p.m.f. $p(x) = 0.1x$ where $x = 1, 2, 3, 4$.Calculate probabilities that aggregate claims equal to $0, 1, 2, 3$. Also, find the probability that aggregate claims exceed 3 units.

2. In a collective risk model, suppose the frequency N of claims is a random variable with possible values $0, 1, 2$ with probabilities $0.4, 0.2, 0.4$ respectively. Amunt X of claims is a random variable with possible values $1000, 3000, 7000$ with probabilities $0.3, 0.4, 0.3$ respectively. Find the distribution of total risk to the company for a short period.

3. For a collective risk model, the number of claims has a negative binomial distribution with parameters $r = 2$ and $p = 1/3$. Claim amounts are mutually independent with two possible values 1 and 2 with equal

chance. Find moment generating function of total claims.

4. Aggregate claim S in a collective risk model has a compound negative binomial distribution, with parameters $k = 100$ and $p = 0.3$ for the negative binomial distribution and individual claim distribution is exponential with mean 100 units. Find mean and variance of S. Using normal approximation, find $P[S > 1.3E(S)]$.

Send the solutions of the exercises at *tuhinsubhra.bh@gmail.com*. **Do the exercises on your workbook clearly and scan those to send to my mail**