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Thermodynamics of asymptotically flat Reissner–Nordstrom black hole

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We established the criteria for thermal stability of a most general black hole in the form of a series of inequalities connecting second-order derivatives of the black hole mass with respect to its parameters. The mass of a black hole depends solely on these parameters, e.g. horizon area and electric charge are these parameters for non-rotating charged black hole. We also introduced the notion of "Quasi stability". It is known how to calculate the fluctuations of these parameters for both stable and quasi stable black holes. In this paper, we consider the simplest black hole having nontrivial parameter, i.e. electrically charged non-rotating asymptotically flat Reissner–Nordstrom black hole (AFRNBH). We will show here that this black hole is not stable anywhere in its parameter space, but it is actually quasi stable, having positive specific heat in some region, violating Hawking's prediction. In fact, this black hole will be shown to exhibit phase transition which is structurally quite different from that in case of Schwarzschild black hole, as predicted first by Hawking. This black hole will also be shown to try to resist its decay under Hawking radiation, but ultimately remains unsuccessful.

Keywords: Quasi stable black hole; black hole thermodynamics; phase transition.

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1. Introduction

It is well known from semiclassical analysis that non-extremal, asymptotically flat black holes are thermally unstable due to decay under Hawking radiation, leading to their specific heat being negative.[1] Sign of specific heat, as a sole criteria, determines the stability of any black hole in semiclassical theory of Hawking. This theory as its drawback treats only matter as quantum entity, spacetime is still classical.[2,3] We had addressed this issue in our earlier works,[4,5] and studied the thermal stability of black holes, from a perspective that is inspired by a definite proposal for quantum spacetime (like Loop Quantum Gravity (LQG),[6,7]) rather than by semiclassical assumptions. We got a series of inequalities, connecting various second-order derivatives of black hole mass with respect to its parameters, as criteria for thermal stability, rather than just a single condition like semiclassical analysis predicted. We still have not studied the equivalence/non-equivalence between this series of stability criteria and positivity of specific heat.

We by virtue of our quantum analysis showed that non-extremal, asymptotically flat black holes do not satisfy all the stability criteria simultaneously.[4] But they do satisfy some of the stability criteria in certain region of parameter space. We call these "quasi stable" black holes in that particular regime of parameter space. In fact AdS black holes, e.g. AdS Kerr-Newman black hole,[8] although thermally stable in certain region of their parameter space, exhibit quasi stability in certain other regions of their parameter space. Thus, quasi stability is an intrinsic nature of the thermal black holes. Quasi stable black holes are shown to have bounded fluctuations for some of their parameters in certain regime of parameter space-like stable black holes.[8–10] But they, due to unfulfilling of all the stability criteria, will ultimately decay under Hawking radiation.

Hawking showed that phase transition was possible in AdS Schwarzschild black hole.[11] It is marked by the sign change of its specific heat that in fact blows up at the point of phase transition. This in terms of fluctuation happens as its area fluctuation starts to diverge from the point of phase transition and keeps on diverging in unstable phase.[4,8–10]

It is interesting to note that Schwarzschild black hole, be it AdS or asymptotically flat, has only one parameter, namely its horizon area. Thus there is only one stability criterion even from our quantum analysis and hence fluctuation of that only parameter is to be considered for this black hole. But quasi stable black holes, essentially having multiple parameters, have multiple numbers of stability criteria. Some of these criteria are definitely violated for them at least regime wise in parameter space. These black holes have individual fluctuations for each of its parameters. Now, it is well known that instabilities in thermodynamics often point to phase transitions. Thus, various instabilities exhibited by quasi stable black holes may be associated with other forms of phase transitions apart from the known Hawking-Page phase transition.[11] We will here look for that in case of Asymptotically Flat Reissner–Nordstrom Black Hole (AFRNBH).

We should obviously look for a black hole with at least one more parameter other than horizon area to study the above in details. Thus, we in this paper consider AFRNBH. The mass of this black hole is related to its electric charge and horizon area in a simple way that various calculations can be done exactly without any approximation. This consequently helps to extract out physics relatively simpler way in an exact manner. AFRNBH, will be shown, is a quasi stable black hole and tries to slow down its decay under Hawking radiation. Its charge fluctuation, determines its electric equilibrium, will be shown to cause phase transition that differs from the Hawking-page phase transition of Schwarzschild black hole. This phase transition, as occurs in decaying black hole, has to be different in nature in comparison with the Hawking-page phase transition. Most interestingly, through this example of AFRNBH, we will explicitly show that positivity of specific heat does not necessary imply the thermodynamic stability of a black hole.

This paper is organized as follows. In Sec. 2, we will recapitulate some of our earlier works briefly for sake of completeness, with emphasis on AFRNBH. We will also show that this black hole is actually quasi stable under Hawking radiation. In Sec. 3, we will calculate the fluctuations for the parameters of AFRNBH. We will also discuss on possible phase transition in details. In Sec. 4, we will compare our obtained results with Hawking's theory and will also discuss over the issue of positivity of specific heat and thermodynamic stability of black hole. Section 5 concludes.

2. Quasi Stability of AFRNBH

The Hilbert space of a generic quantum spacetime is product of bulk and boundary space. Thus, a generic quantum state on this Hilbert space is tensor product of bulk and boundary state. Now, the full Hamiltonian operator, operating on this Hilbert space, has two parts. One part is the bulk Hamiltonian and another is the boundary Hamiltonian. Similarly, generic charge operator, operating on this Hilbert space, has two parts; one is bulk part and another is boundary part. The first class constraints are realized on this Hilbert space as annihilation constraints on the quantum states. The bulk Hamiltonian operator thus annihilates bulk physical states. Similarly, Gauss's constraint of electrodynamics is realized by the annihilation of the bulk physical states by the bulk charge operator. The mathematical details of these facts are given in Ref. 4.

Any charged black hole has discrete values of horizon area and electric charge. This is expected in any quantum theory of gravity, e.g. Loop Quantum Gravity supports this.[6] Now, we can consider a charged black hole to be immersed in a heat bath, with which it can exchange energy and electric charge (Q). Thus, we can write down the grand canonical partition function (Z_G) as summation over possible eigenstates with appropriate weightage.[12] These states are generic quantum states on the full Hilbert space. But the first class constraint and Gauss's constraint reduce the summation to only summation over the boundary states. Thus generic grand canonical partition function becomes the same over the boundary only.[4] Thus, we do not require to know the internal statistical components of black hole at all, considering horizon of the black hole to be the boundary of spacetime, to calculate the full grand canonical partition function of the black hole. We can convert this summation, with the help of Poisson's resummation formula,[13] into integration and determine the criteria for thermal stability. An electrically charged black hole, in thermal equilibrium, is represented by the saddle point ($\bar{A}$, $\bar{Q}$). $\bar{A}$ denotes horizon area (A) at equilibrium and so on. It is shown earlier[4] that this partition function turned out to be integration over the space of fluctuations $a = (A - \bar{A})$, $q = (Q - \bar{Q})$ around the saddle point and is given as[4]

$$Z_G \approx \int da\, dq \exp\left(-\frac{\beta}{2}[(M_{AA})a^2 + (M_{QQ})q^2 + (2M_{AQ})aq]\right). \quad (1)$$

The only assumption we have made is that the mass of the black hole (M) is a function of its electric charge and horizon area. Here, $M_{AA} \equiv \frac{\partial^2 M}{\partial A^2}$, $M_{AQ} \equiv \frac{\partial^2 M}{\partial A \partial Q}$, etc. and they are evaluated at the saddle point. Thus, the above expression of grand canonical partition function, although may have some similarities with semi classical analysis, is constructed entirely from the consideration of quantum geometry of the black hole. $S(A)$ is the entropy of the black hole and is taken to be equal to $\frac{A}{4A_P}$, where A_P is the Planck area.[2] We take $4A_P$ equals to unity for sake of simplicity and can bring it back just by dimensional analysis.

The convexity of the above integral (1), with the help of calculations done in Appendix A, leads to the criteria for thermal stability of the black hole and are either given as

(i) $M_{QQ} > 0$, $\mathrm{Sig}\left(\frac{(M_{QQ}M_{AA} - (M_{AQ})^2)}{M_{QQ}}\right)$ is positive, or

(ii) $M_{AA} > 0$, $\mathrm{Sig}\left(\frac{(M_{QQ}M_{AA} - (M_{AQ})^2)}{M_{AA}}\right)$ is positive. This is exactly equivalent to what we obtained earlier.[4]

Here $\mathrm{Sig}\left(\frac{(M_{QQ}M_{AA} - (M_{AQ})^2)}{M_{AA}}\right)$ denotes the sign of the function $\frac{(M_{QQ}M_{AA} - (M_{AQ})^2)}{M_{AA}}$. We will see soon that this way of writing the stability criteria is very helpful for analyzing quasi stable black holes. It is important to note that these stability criteria are completely based on possible quantum mechanical nature of the spacetime and generic as well as we have not assumed any special property of the black hole, except the fact that its mass is an arbitrary function of its area and charge. This arbitrariness makes the above stability criteria applicable to so called classical black holes as well. In fact the relationships among the mass and various parameters of the black holes are required only to test the stability of a particular black hole, but not to derive the generic stability criteria at all. Hawking in his case by case approach did not give any general prescription for stability criteria of an arbitrary black hole. In that formalism, one has to know the global structure of spacetime to evaluate the path integral.[3] But our formalism requires only the knowledge of the structure of the horizon. In fact in our formalism, we do not have to treat black holes case by case to obtain stability criteria. We obtained a series of stability criteria for a generic black hole, rather than just a single criteria of positivity of specific heat. In fact, positivity of specific heat can at most be one of the series of criteria. We will see in this paper that in certain regime of parameter space specific heat of AFRNBH hole is positive, although it is nowhere thermodynamically stable.

We have realistically assumed that (inverse) temperature β $\left(\equiv \frac{S_A}{M_A}\right)$ is positive.

The mass (M) of AFRNBH depends on its parameters as[14]

$$M = \frac{\sqrt{A}}{4\sqrt{\pi}} + \frac{\sqrt{\pi}Q^2}{\sqrt{A}}. \tag{2}$$

We can now calculate the temperature of AFRNBH from Eq. (2) and it will be function of its electric charge (Q) and horizon area (A). On calculation, it turns out that temperature $(T) = \frac{1}{8\sqrt{\pi A}}\left(1 - \frac{4\pi Q^2}{A}\right)$ and hence is positive only if $\frac{Q^2}{A} < \frac{1}{4\pi}$. This restricts the parameter space. At absolute zero temperature, i.e. $T = 0$, black hole is thermodynamically inert in the sense that it stops interacting with its surrounding. Thus, positivity of temperature, i.e. $T > 0$ is an essential criteria for thermal black holes to interact with its surrounding and execute their thermodynamic properties. Hence, we will not study extremal RN black hole here as it would not be interesting from thermodynamic perspective. It is true that the above relation (2) is classical and it is expected that this relation may acquire some corrections if one wants to derive the same from quantum mechanical perspective. But this is not something that we want to derive in this paper. We instead want to analyze the classical relation in light of our stability criteria. We find that even this gives new results for the thermal stability, in respect to what we have known so far.

We can calculate various second derivatives of the black hole mass (M) with respect to its parameters from Eq. (2). On calculation, this turns out that

$$M_{QQ} = \frac{2\sqrt{\pi}}{\sqrt{A}}, \quad M_{AQ} = -\frac{\sqrt{\pi}Q}{A^{3/2}}, \quad M_{AA} = -\frac{1}{16\sqrt{\pi}A^{3/2}} + \frac{3\sqrt{\pi}Q^2}{4A^{5/2}},$$

$$(M_{QQ}M_{AA} - (M_{AQ})^2) = \left(-\frac{1}{8A^2} + \frac{\pi Q^2}{2A^3}\right).$$

Thus $(M_{QQ}M_{AA} - (M_{AQ})^2)$ is positive only if $\frac{Q^2}{A} > \frac{1}{4\pi}$. But this region of parameter space is not accessible to any real AFRNBH as it is excluded due to positivity of temperature. Hence $(M_{QQ}M_{AA} - (M_{AQ})^2)$ is negative through out its physically accessible regime of parameter space. Now, M_{QQ} is always positive while M_{AA} is negative if $\frac{Q^2}{A} < \frac{1}{12\pi}$. Thus, AFRNBH can never be thermally stable as it never satisfies stability criteria completely. So,[9] AFRNBH is actually a quasi stable black hole.

3. Fluctuation and Phase Transition of AFRNBH

We have earlier shown that[9] quasi stable black holes possess bounded fluctuations for some of its parameters in certain regions of parameter space. So, same is expected in case of AFRNBH. $\Delta(A)^2$ measures the fluctuation of horizon area from its equilibrium value. It is mathematically expressed, for non-rotating charged black hole, as,[8,9] $\Delta(A)^2 \equiv \frac{\int da\, dq\, a^2 f(a,q)}{\int da\, dq \inf(a,q)}$; where $f(a,q) = \exp\big(-\frac{\beta}{2}[(M_{AA})a^2 + (M_{QQ})q^2 + (2M_{AQ})aq]\big)$. Similarly, $\Delta(Q)^2$ can be defined. We, with the help of calculations done in Appendix A, can determine the regions of convergences of $\Delta(A)^2$ and $\Delta(Q)^2$, if exist. Now $(M_{QQ}M_{AA} - (M_{AQ})^2)$ is always negative for AFRNBH.

Keeping this in mind, we can conclude that

(1) $\Delta(A)^2$ always blows up as M_{QQ} is always positive.
(2) $\Delta(Q)^2$ converges and converges to the value $\frac{M_{AA}}{2\beta(M_{QQ}M_{AA}-(M_{AQ})^2)}$ only if M_{AA} is negative, i.e. $\frac{Q^2}{A} < \frac{1}{12\pi}$.

Of course it is true that AFRNBH will ultimately decay. It will gradually become smaller and smaller in size as fluctuation of its area is unbounded. Thus, even if the ratio $\frac{Q^2}{A}$ is lesser than $\frac{1}{12\pi}$ at the beginning, it will increase as area (A) will decrease. But this ratio cannot be greater than $\frac{1}{4\pi}$. In the region $\frac{1}{4\pi} > \frac{Q^2}{A} > \frac{1}{12\pi}$, electric charge ($Q$) of this black hole fluctuates appreciably and this fluctuation in turn reduces the value of Q. Thus, this ratio will once again be smaller than $\frac{1}{12\pi}$. Now, again the same thing will happen, i.e. the value of $\frac{Q^2}{A}$ will again increase and then it will again decrease. This will go on. The black hole will continue to lose its area and electric charge. Thus, it heads towards a black hole with certain minimum area,[15] having almost no electric charge. It is true that charged particles are also emitted in parallel to Hawking radiation from a charged black hole by the mechanism of Schwinger emission of charged particles.[16,17] We here consider only thermodynamical fluctuation of electric charge to show that discharging happens as we are considering thermodynamics of black hole solely.

Now consider the case of AdS Schwarzschild black hole. It is well known that this black hole is thermally stable only if $A > l^2$, l is the cosmic length.[11] The only criteria for thermal stability of this black hole is the positivity of M_{AA} and it holds only in the region $A > l^2$. The specific heat of this black hole, inversely proportional to M_{AA}, is proportional to fluctuation of its area ($\Delta(A)^2$) as long as M_{AA} is positive. $\Delta(A)^2$ starts to diverge when M_{AA} becomes zero and keeps diverging for negative values of M_{AA}. Similarly, specific heat diverges when M_{AA} equals to zero and becomes negative for negative values of M_{AA}. Thus, we see that M_{AA} equals to zero is the point of phase transition. So, specific heat changes discontinuously during phase transition. But fluctuation in horizon area, although blows up, does not show any discontinuity. It is very a well-known fact that fluctuations of the parameters of a system determine the stability and consequently the phase transitions of that system. Thus, we can conclude that divergence in fluctuation of any parameter of the system indicates the phase transition of that system.

We will now focus on the black hole that we are considering in this paper, i.e. AFRNBH. The mass of this black hole depends on two parameters, its horizon area and electric charge. So, we have to take into account the fluctuations of both these parameters individually to consider phase transitions. Now, $\Delta(A)^2$ always diverges and consequently this black hole cannot resist itself from decaying away under Hawking radiation. But $\Delta(Q)^2$ does not always diverge and hence phase transition can occur. We have already seen that M_{AA} equals to zero marks the divergence of $\Delta(Q)^2$ and remains diverging as long as M_{AA} is positive. Thus, phase transition occurs when M_{AA} becomes equal to zero. It is interesting to note that $\Delta(Q)^2$, evident from its expression, tends to become zero as M_{AA} approaches to zero, from negative side and it then suddenly starts to diverge as M_{AA} becomes zero and positive onwards. Thus, fluctuation of electric charge diverges discontinuously at the point of phase transition, unlike the continuous divergence of area fluctuation in Hawking-Page phase transition. In fact, fluctuation in electric charge is proportional to the electric capacitance of the black hole as long as the fluctuation is converging.[18] Electric capacitance of a black hole is defined and is given as, $S_Q \equiv \beta \cdot \partial Q/\partial\bar{\Phi}$ and has been shown to be equal to $\beta \cdot \Delta(Q)^2$.[18] Here $\bar{\Phi}$ is β times the electric potential of the charged black hole. Thus, S_Q equals to $M_{AA}/(2(M_{QQ}M_{AA} - (M_{AQ})^2))$. So, electric capacitance changes its sign from positive to negative as the black hole changes its phase from stable to unstable, passing through its zero value. Hence, electric capacitance changes its sign smoothly during phase transition, unlike the abrupt change of specific heat in case of AdS Schwarzschild black hole. Thus, we find an explicit difference in the structure of phase transition(s) between black holes having multiple parameters with black holes having single parameter, i.e. horizon area.

The decay of any black hole is approximately governed by Stefan–Boltzmann law as the profile of black hole radiation is approximately same as that of a black body. So, luminosity (L), the power radiated per unit surface area, varies with its temperature (T) as $L \propto T^4$. Temperature is a function of electric charge and horizon area and hence any fluctuation in them would make temperature fluctuating as well. This in turn makes luminosity fluctuating too. Thus, the fluctuation in luminosity is given as

$$\Delta L \propto \left(\frac{\partial T}{\partial A}\Delta A + \frac{\partial T}{\partial Q}\Delta Q\right).$$

Now, $T \propto M_A$ and hence

$$\frac{\partial T}{\partial A} \propto M_{AA}, \quad \frac{\partial T}{\partial Q} \propto M_{AQ}, \quad \therefore\ \Delta L \propto (M_{AA}\Delta A + M_{AQ}\Delta Q).$$

Thus, we see the expression within the above parenthesis determines the sign of ΔL, i.e. whether decay rate would increase or not. Now, ΔA is always large negative and M_{AQ} is always negative. Thus, in the regime $\frac{Q^2}{A} > \frac{1}{12\pi}$, M_{AA} is positive and ΔQ is large negative. Hence, the term $M_{AA}\Delta A$ is negative in this particular regime and consequently tries to slow down the decay process. But the term $M_{AQ}\Delta Q$ is positive and hence tries to fasten the decay process, compatible with the fact that black hole is still electrically unstable. Thus, a competition goes on between these two terms. This is certainly absent in case of unstable black hole, e.g. asymptotically flat Schwarzschild black hole (AFSBH). The term $M_{AA}\Delta A$ becomes positive in the regime $\frac{Q^2}{A} < \frac{1}{12\pi}$, but the term $M_{AQ}\Delta Q$, although positive, is substantially tiny as electric charge does not fluctuate much. This is again compatible with the fact that black hole is electrically stable in the region $\frac{Q^2}{A} < \frac{1}{12\pi}$. Thus, there is at least some tendency of AFRNBH to reduce its decay rate under Hawking radiation. This tendency makes it different from any other unstable black hole.

It is of course true that decay rate of any black hole cannot be calculated without considering the full dynamics of that black hole. We have to consider the full-fledged theory of quantum gravity for that. But such calculations are not done yet in any non-perturbative theory of quantum gravity. But we can still qualitatively conclude whether a black would enhance its decay rate in certain regime of parameter space or not. We have done so just by considering the thermodynamical aspect of the black hole only. This is something robust and makes our analysis unique.

4. Comparison with Hawking's Semi Classical Theory

Hawking is the first person to discover black hole radiation.[2] He in fact also predicted the thermal instability of AFSBH due to its negative specific heat.[11] He also showed that specific heat blowed up at phase transition. In fact in his case-by-case study approach, he used the sign of specific heat as the only criteria to determine the thermal stability.[3] We will show explicitly that this is not the case here at all.

Now, specific heat (C) is defined as

$$C \equiv \frac{\partial M}{\partial T}.$$

Since both M and T are function of Q and A. Thus, we can write

$$C = \frac{\partial M}{\partial A} \cdot \frac{\partial A}{\partial T} + \frac{\partial M}{\partial Q} \cdot \frac{\partial Q}{\partial T}.$$

On calculation, we find that

$$C = -8\left(\frac{1}{8\pi} - \frac{Q^2}{A}\right) \Big/ 3\left(\frac{1}{12\pi} - \frac{Q^2}{A}\right).$$

Following the above expression, we get

(1) $C < 0$ if either $\frac{Q^2}{A} < \frac{1}{12\pi}$ or $\frac{Q^2}{A} > \frac{1}{8\pi}$,

(2) $C > 0$ if $\frac{1}{12\pi} < \frac{Q^2}{A} < \frac{1}{8\pi}$.

So, according to Hawking's criteria, in the region $\frac{1}{12\pi} < \frac{Q^2}{A} < \frac{1}{8\pi}$, AFRNBH is thermally stable as specific heat is positive. But we have already shown explicitly that this black hole can not be stable any where in its accessible parameter space. Thus, we see explicitly that the series of stability criteria, derived by us earlier, are not equivalent with the condition of positivity of specific heat, derived first by Hawking. It is interesting to note that specific heat in fact changes its sign for AFRNBH, while it is nowhere stable within its accessible regime of parameter space. Our series of thermal stability criteria hold good for Schwarzschild black holes (both asymptotically flat and AdS) as well and there it matches with Hawking's prediction as well. Thus when the number of parameter is one, namely, horizon area, both the theory predicts same. This is as in those cases, our stability criteria is essentially the criteria of positivity of specific heat, as predicted by Hawking. But mismatches start when a black hole possesses multiple parameters, i.e. parameters in addition to its horizon area, e.g. both rotating and non-rotating charged black holes. This implies that black holes with a single parameter (on which its mass depends) is the region of common prediction. In other words, this common region is the limit where our prediction boils down to that of Hawking. The presence of this common limiting region is quite expected as otherwise it will be meaningless to have a improved version of thermal stability criteria for black holes over the existing version of stability criteria as predicted by Hawking. These evidently put the limitations on the applicability of Hawking's criteria over thermal stability of a black hole in the sense that positivity of specific heat cannot be the sole criteria for thermal stability of a generic black hole. In his semi classical theory, Hawking treated the black holes classically. But we derived the stability criteria by treating the spacetime quantum mechanically. Thus, we can expect even theoretically that our result should mismatch with Hawking's criteria at some point of time. In this paper we have shown this explicitly in case of AFRNBH.

5. Discussion

AFRNBH, like unstable black holes, ultimately decays under Hawking radiation. But its electric charge almost does not fluctuate in certain region of its parameter space. It tries to resist its decay under Hawking radiation. This feature is somewhat similar to that of a stable black hole. Thus, it possesses some sort of dual property. Our analysis holds for macroscopic black holes. But close to end state of a black hole, we have to solve the complicated Hamiltonian constraint. But it is true that there would be some remnant of a black hole at its end state, with a minimum horizon area according to theory like LQG. This fact has been mentioned in Ref. 15 in which it is concluded that these remnants could form component of dark matter as well. In this sense, our analysis may have some impacts on dark matter physics.

We have seen that thermodynamic phase transitions, apart from known Hawking-Page phase transition, exist in quasi stable black holes. Electrical phase transition in AFRNBH is example such phase transition. This phase transition occurs multiple times during its decay. The usual phase transition is characterized by the discontinuous sign change in specific heat. But the phase transition for AFRNBH is characterized by the continuous sign change in its electric capacitance. This makes the phase transition new and interesting. In fact thermodynamic phase transition for Kerr-Newman black hole is expected to be even more interesting in presence of its rotation. We will not although advance regarding that in this paper.

Appendix A. Calculations for Integrations

Let $I_1 = \int\int dx\,dy \exp(-(ax^2 + by^2 + 2cxy))$. We can now write $(ax^2 + by^2 + 2cxy)$ as $\left(a(x + cy/a)^2 + \frac{(ab-c^2)y^2}{a}\right)$. We redefine the variables x, y as

$$\begin{pmatrix} \underline{x} \\ \underline{y} \end{pmatrix} = \begin{pmatrix} 1 & \dfrac{c}{a} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{A.1}$$

In terms of the variables $\underline{x}$, $\underline{y}$, we can rewrite I_1 as

$$I_1 = \int d\underline{x} \exp(-a\underline{x}^2) \cdot \int d\underline{y} \exp\left(-\frac{(ab-c^2)\underline{y}^2}{a}\right)$$

and hence this integral is expressed as products of two independent integrals. Here, Jacobian factor due the transformation of variables is identically unity. This integral converges only when both a and $(ab-c^2)$ are positive.

Now consider the integral $I_2 = \frac{\iint dx\,dy \cdot y^2 \exp(-(ax^2+by^2+2cxy))}{\iint dx\,dy \cdot \exp(-(ax^2+by^2+2cxy))}$ and it converges, using the above technique, only if $\frac{(ab-c^2)}{a}$ is positive, i.e. $(ab-c^2)$ and a have same sign. Thus I_2 converges even if a is negative, unlike I_1. I_2, if converges, is equal to $\frac{a}{2(ab-c^2)}$. Similarly $\frac{\iint dx\,dy \cdot x^2 \exp(-(ax^2+by^2+2cxy))}{\iint dx\,dy \cdot \exp(-(ax^2+by^2+2cxy))}$ converges only if $\frac{(ab-c^2)}{b}$ is positive, i.e. $(ab-c^2)$ and b have same sign. The last integral and I_2, respectively diverges for $b \geq 0$ and $a \geq 0$ if $(ab-c^2)$ is negative.

We have already expressed I_1 as product of two separate integrals and hence we can conclude from the details of I_1 and I_2 that $\frac{\int dx \cdot x^2 \exp(-ax^2)}{\int dx \cdot \exp(-ax^2)}$ converges and equals to $\frac{1}{2a}$ if and only if $a > 0$; otherwise it diverges. All the integration variables considered here run throughout the real line and integrations are taken for the whole range. Here, the constants a, b, c are real quantities.

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