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Optimizing a supply chain problem with nonlinear penalty costs for early and late delivery under generalized lead time distribution

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ABSTRACT

This article presents a tangible vendor–buyer cooperative strategy that benefits both, the vendor and the buyer where the demand is deterministic constant, and the delivery lead time follows a general distribution. To build a realistic coordination mechanism, a delivery tolerance time range is specified beyond which two different types of nonlinear penalty costs termed as an *early delivery penalty cost* and *late delivery penalty cost* are assessed. Shortages are allowed for a short time-span. The penalty costs are taken as a product of a linear function of delivery lead time and a nonlinear function of the delivery lot size. The problem is formulated as a multi-variable mixed-integer nonlinear programming (MINLP) problem and the objective of this research is to achieve the minimum integrated expected cost where decision variables are: reorder point, delivery lot size, number of deliveries, and delivery time thresholds. Since closed-form solutions are not immediately obtained, different search procedures are employed to resolve issues relating to an integer solution. Numerical results are provided for uniform, exponential and normal distributions of delivery lead time to establish the general model.

1. Introduction

The efficiency of a supply chain network is greatly influenced by the reliability of the supply process. The success of a supply chain lies beneath the proper timing of delivery of goods to the intermediate parties. This research work has adopted the integrated vendor–buyer optimization policy together with the idea of generalized lead time and nonlinear penalty cost for early and late delivery of shipments. This research contributes an improved delivery timing strategy to enhance over all supply chain performance. When a lot is delivered within a delivery tolerance period, then no penalty cost is assessed to the vendor, which gives latitude to both the vendor and the buyer to cope with the uncertainty of delivery mechanism and transportation time.

A real-world example of the investigated problem is the motivation of this problem under consideration and it is explained here. In Asian countries, the labor charge is significantly low for which the production cost also becomes low. Many industries such as textile, electronic goods in diverse Asian countries manufacture their product at a low price and export them to Europe, America and Australia. They prefer the naval route as the transportation mode for transporting their products to overseas due to low transportation cost. Although the transportation cost in a naval route is low; but the chance of delivery delay is high which may cause significant loss to the buyer (importer). The buyer incurs the expense for keeping the storehouse ready before receiving the ordered lot and loses both market goodwill and the potential profit during market peak time if they are not delivered in time. These losses are increased with an increasing number of units. Another practical situation in such import–export business is also explained here. Usually, the large-scale industries such as textile industries, steel industries, electronic goods industries and automobile industries maintain their owned or rented warehouses. They bear the expense to maintain the infrastructure and good storage environment (temperature, humidity, etc.) of the warehouses. If the ordered lot reaches too early, then sometimes a space problem may arise. Such space problem is increased with an increase in the number of units. To keep the early delivery lot, the buyers must rent another storehouse and bear the burden of maintaining it, which increases the storage cost, and hence the total cost. To discourage the practice of such a costly early and late delivery, the buyer imposes a significantly high penalty charge to the vendor for both early and late arrival of ordered items. Such types of penalty cost scenarios are usually found in international business of garments and other luxury goods.

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Plenty of research works are available including a lead-time consideration. A brief literature survey is given in the following subsection.

1.1. The literature

Realizing the necessity of compact coordination in a supply chain system, a significant number of researchers have incorporated several papers in multi-dimensional directions including variable lead time. Pan and Yang (2002) have developed a supply chain model with controllable lead time and stochastic lead time demand. Ouyang et al. (2004) have extended the work of Pan and Yang (2002) by introducing the concept of stochastic demand in conjunction with controllable lead time while Hoque and Goyal (2006) have extended Pan and Yang (2002) model by incorporating the shipments of equal or unequal sized batches together with controllable lead time. A reduction function for lead time is considered by Hsu and Huang (2009) whereas Li et al. (2012) have developed a supply chain coordination system for multi-product with controllable lead time. Braglia et al. (2014) have investigated a two-stage supply chain with a safety stock management and consignment stock agreement where they have considered the demand and lead time both are random in nature. An operational consignment stock policy for normally distributed demand is stated by Yi and Sarker (2014) where buyers' space limitation and controllable lead time are considered. Besides these, many articles are incorporated by considering fixed as well as variable lead time. A few of them are Glock (2012), Yi and Sarker (2013), Rodrigues and Yoneyama (2020), and Das Roy and Sana (2021).

In management and other scientific researches, the normal distribution is the most commonly used probability distribution among the others. Several researchers have developed inventory models with the consideration of normal demand. Dey and Chakraborty (2012) have framed an inventory model where the demand rate is assumed to be a normally distributed fuzzy random variable. They have considered both non-truncated and the truncated normal distributions of demand. The concept of truncated normal distribution is also discussed by Thomopoulos (2015). Hossain et al. (2017) have considered a supply chain system with a non-truncated normal distribution of lead time while an integrated supply chain model with normally distributed lead time demand is investigated by Das Roy and Sana (2020). The present study has discussed three types of lead time distributions. Non-truncated normal distribution of lead time is one of them.

The occurrence of a stock-out situation in an inventory management system is very common. Shortages can be backlogged in two ways: Partially or completely. Many authors [Ng et al. (2001), Das Roy et al. (2012, 2014), Hossain et al. (2017), San-Joséa et al. (2019)] have included backlogging in their studies. The proposed article has also considered backlogging. Any article that has addressed the concept of nonlinear early delivery penalty cost together with nonlinear late delivery penalty cost in a supply chain having random delivery lead time seldom follows a general distribution. Recently, researchers have focused on the consequences when the lead time is stochastic in nature and follows some known probability distributions. Lee et al. (2007) have investigated an integrated inventory model with stochastic lead time, ordering cost reduction and backorder discount whereas a multi-supplier and single buyer supply chain coordination system with a milk-run delivery network is presented by Zhou et al. (2012). They have included stochastic lead time and capacity constraints in their study. A supply chain model with stochastic lead time is also discussed by Lin (2016) and Hossain et al. (2017).

The concept of penalty cost for delivery lateness is addressed by many authors. Guiffrida and Jaber (2008) have introduced penalty cost for early and late delivery in a supply chain to study the managerial and economic impacts of reducing delivery variance while an optimal position of supply chain delivery window which minimizes the expected penalty cost for delivery earliness and lateness is determined by Bushuev and Guiffrida (2012). Zhu (2015) has incorporated a decentralized supply chain where the penalty cost is considered in terms of compensation to the customer for delivery lateness. Hossain et al. (2017) have discussed a vendor–buyer cooperative policy in an integrated supply chain model with a general distribution of lead time. They have introduced the concept of penalty cost for delivery lateness. A cost base delivery performance model is developed by Bushuev (2018) where he has considered an expected penalty cost for delivery earliness and tardiness. The present study introduces nonlinear penalty costs for early delivery as well as for late delivery of the ordered lot size. Biswas and Sarker (2020) have developed an operational planning of supply chains in a production and distribution center with just-in-time delivery policy. Lin et al. (2021) have showed how to reduce optimally the setup cost and lot size for economic production quantity model with imperfect quality and quantity discounts. A comparison between the contributions of previous works with the present study is presented in a tabular form (see Table 1).

Flexibility in lead time plays an important role in operating a coordinated system within a certain tolerance time (early or late arrivals of shipments). From Table 1, the contribution of the present paper in the literature and the comparison of the proposed research with other articles is clearly observed. Two main points are highlighted in this respect. First, most of the researchers have restricted their study by considering lead time as a deterministic variable or random variable which follows a specific probability distribution. Consideration of general distribution of lead time is rarely observed in those studies which is one of the features of this study. Secondly, very few authors have included a penalty cost for early or late delivery or both. Also, the researchers who have included penalty cost for early or late delivery, have considered linear penalty cost. In the present paper, a nonlinear penalty cost has been addressed to generalize the penalty function. The method also leaves another aspect of the contributions to capture other variants of such cost function as the system subscribes to fit the existing system.

1.2. Definition and objective of the problem

In the present study, an integrated vendor-buyer cooperative supply chain network is proposed where the lead time is assumed to be generally distributed. The buyer provides a delivery tolerance range to the vendor. If the vendor delivers the ordered lot beyond this delivery tolerance range, he must face two types of penalty costs: *early delivery penalty cost* and *late delivery penalty cost*. If the delivery lot reaches before the lower limit of a delivery tolerance range, then it increases the holding cost of the buyer. The buyer charges a penalty cost termed as *early delivery penalty cost* to the vendor equivalence to this extra holding cost. Again, if the delivery lot reaches after the upper limit of the delivery tolerance period, then the vendor will be responsible for paying a penalty cost to the buyer. Here, this type of penalty cost is termed as *late delivery penalty cost* which is equivalent to the buyer's loss of market goodwill and opportunity loss for unable to sell the product during this late period. The main goal of the research work is to determine the optimal values of the replenishment lot size, reorder point, number of shipments for minimizing the integrated expected cost under the environment of generalized lead time distribution of delivery to improve the supply chain performance.

The entire paper is organized into seven sections. Section 1 carries the introduction part. The notation and assumptions of this study are stated in Section 2. Section 3 describes the general model while Section 4 illustrates the solution procedure. Numerical results are provided in Section 5 and sensitivity analyses are carried out in Section 6. Section 7 presents the conclusion of the whole study.

2. Notation and assumptions

The notation and assumptions used in the present paper to develop the model are as follows.

Table 1
Contribution of different authors in the related field.

Author (s)	Supply chain	Safety stock/reorder point	Generalized lead time	Shortage	Penalty cost	
					Early delivery	Late delivery
Ng et al. (2001)	✓	✓		✓		
Pan and Yang (2002)	✓	✓				
Ouyang et al. (2004)	✓	✓		✓		
Hoque and Goyal (2006)	✓	✓				
Lee et al. (2007)	✓	✓		✓		
Guiffrida and Jaber (2008)	✓				Linear	Linear
Hsu and Huang (2009)	✓	✓		✓		
Li et al. (2012)	✓	✓				
Glock (2012)	✓	✓		✓		
Das Roy, Sana, and Chaudhuri (2012)	✓			✓		
Zhou et al. (2012)	✓	✓		✓		
Bushuev and Guiffrida (2012)	✓				Linear	Linear
Braglia et al. (2014)	✓	✓		✓		
Yi and Sarker (2014)	✓	✓		✓		
Zhu (2015)	✓					Linear
Lin (2016)	✓	✓		✓		
Hossain et al. (2017)	✓	✓	✓	✓		Linear
Bushuev (2018)	✓				Linear	Linear
San-Jose et al. (2019)		✓		✓		
Das Roy and Sana (2020)	✓	✓		✓		
Das Roy and Sana (2021)	✓	✓		✓		
Present paper	✓	✓	✓	✓	Nonlinear	Nonlinear

2.1. *Notation*

The notation used to describe the model is as follows:

(a) *Common notation*

D	Annual demand (units/year).
τ	Length of lead time to deliver the replenishment lot (year).
$f(\tau)$	The probability density function for the lead time τ.
L	Upper bound or maximum length of the lead time (year) after placing an order.
l	Lower bound or minimum length of the lead time (year) after placing an order.
d_E	Early delivery tolerance factor of the buyer, where $0 < d_E < 1$.
d_L	Late delivery tolerance factor of the buyer, where $d_L > 1$.
t_E	Lower limit or minimum delivery tolerance period (year) after placing an order, where $t_E > l$.
t_F	Upper limit or maximum delivery tolerance period (year) after placing an order, where $t_F < L$.

(b) *Notation for the vendor*

m	The nonlinearity factor, $0 < m < 1$.
C_V^v	Set up cost for the vendor ($/setup).
C_H^v	Stock holding cost for the vendor ($/unit/year).
C_E^v	Early delivery penalty cost of the vendor if the delivery lot arrives before the delivery tolerance period of the buyer ($/unit/year).
C_F^v	Late delivery penalty cost charged to the vendor if the delivery lot arrives after the delivery tolerance period of the buyer ($/unit/year).
EAC_V	The expected cost of the vendor ($/year).

(c) *Notation for the buyers*

C_0^b	Ordering cost for the buyer ($/order).
C_h^b	Stock holding cost for the buyer ($/unit/year).
C_b^b	Backlogging cost for the buyer ($/unit/year).
T	Cycle time of the buyer (year).
EAC_B	The expected cost of the buyer ($/year).

(d) *Decision variables*

Q	The order lot size for the buyer (units/order).
R	Reorder point of the buyer (units).
n	The number of deliveries from the vendor to the buyer in a replenishment cycle of the vendor.
$EACI$	The integrated expected cost function ($/year), where $EACI = EACI(Q, R, n)$.

2.2. *Assumptions*

The assumptions used to describe the system under consideration are as follows:

(1) The supply chain system consists of a single vendor and a single buyer.
(2) Demand of product is deterministic and known.
(3) The vendor produces the ordered lot nQ and delivers them into n number of shipments of fixed lot-size Q.
(4) Lead time to replenish vendor's warehouse is zero.
(5) Lead time to replenish buyer's order is stochastic in nature.
(6) The appropriate inventory level is not permitted to fall below the reorder point just after a replenishment occurs (see Fig. 3). There is one outstanding order.
(7) The buyer provides a delivery tolerance range $[t_E, t_F]$ to the vendor, where t_E and t_F indicate the length of lower and upper delivery tolerance periods after placing an order respectively (see Fig. 1 and Fig. 3).

There is an agreement between the vendor and the buyer that if the replenishment lot arrives before the lower delivery tolerance period t_E of the contracted delivery tolerance range $[t_E, t_F]$, then the buyer charges a penalty cost termed as *early delivery penalty cost* to the vendor which will be equivalent to the excessive holding cost that the buyer has to bear for storing the lot for an extra time. Generally, holding cost increases with stock and time. Therefore, the *early delivery penalty cost* is considered as a function of ordered lot size and the extra time-span before t_E. This additional cost is assessed to the vendor and it does not affect the buyer [see Theorem 1]. Again, it is assumed that if the delivery lot arrives to

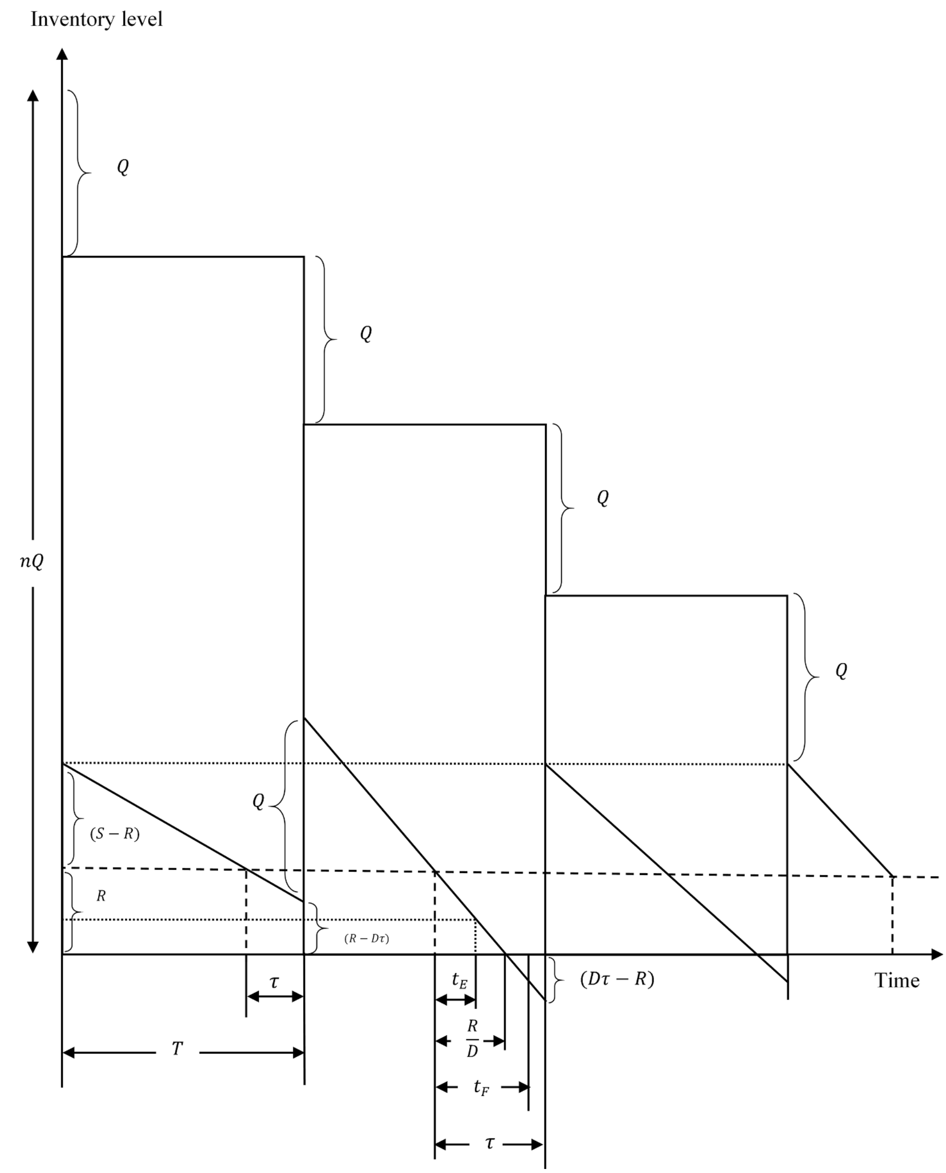

Fig. 1. Inventory level of the vendor and buyer over time.

the buyer premises after the upper bound (t_F) of the delivery tolerance range $[t_E, t_F]$, then the vendor has to pay a penalty cost to the buyer as the buyer loses his potential profit during this time-span. This penalty cost is termed as *late delivery penalty cost* and is equivalent to the sum of goodwill loss and opportunity loss. Now, the loss of market goodwill and opportunity loss increases with an increase in the number of units unable to sell during the late period. Therefore, the *late delivery penalty cost* is assumed as a function of ordered lot size and the time-span after t_F. This penalty cost also affects the vendor but not the buyer [see Theorem 1]. Clearly, if the delivery lot arrives within the delivery tolerance periods, then the vendor does not face any penalty cost.

(8) The two distinct delivery tolerance periods t_E and t_F are used as delivery tolerance thresholds to obtain the delivery tolerance range in terms of reordering time.

(9) Shortages are permitted at the buyer's level [see Roy et al. (2010), Das Roy, Sana, and Chaudhuri (2011a, b), Glock (2012), Das Roy and Sana (2017), Hossain et al. (2017)]. If they occur, then they are completely backorder in the next ordering cycle. The vendor does not face any stock-out situation to deliver the replenishment lot to the buyer because of the vendor's zero lead-time replenishment.

(10) The system runs under a fixed time horizon, say T, that repeats itself over time to keep the continuity of operations.

Unless the parties are cooperative, the system cannot work well and, in our opinion, they must do it for mutual benefits—the vendor is expected to comply more tolerantly with the manufacturer/seller to earn goodwill and to keep the active business deal. In order to material such an agreement, the manufacturer/seller must have access to the vendor's transportation data or it may collect these data at their own initiative as to the expected lower and upper delivery tolerance periods t_E and t_F and their corresponding to penalty costs so that they can mutually plan to work in an agreeable optimal arrangement.

3. The general model

In this section, the model is described and framed mathematically.

3.1. The model description

It will be most cost-effective if the vendor optimizes its own inventory replenishment because if the vendor optimizes its own inventory replenishment, it does consider its own interests first leaving aside the other's interest for minimizing the vendor's cost most. Hence, the vendor's cost will be minimized independently benefitting himself under this assumption. Also, if the vendor produces nQ units and ships only Q units to a buyer into n times, then the vendor can save his/her expensive setup cost. On the other hand, if he has to consider other's/ buyer's interest, he must incorporate the associated costs of the buyer as well under a common or joint model. Thus, the joint cost model will have theoretically a tradeoff between the costs of both vendor and buyer, and obviously the joint cost will be minimized at mutually agreed order quantity. It may be worth mentioning that many previous works also assumed such a situation [see Golhar and Sarker (1992), Sarker and Parija (1996), Hsu and Huang (2009), Yu et al. (2011), Das Roy, Sana, and Chaudhuri (2012), and Lin (2013)].

Suppose there is an agreement between the vendor and the buyer that the vendor will deliver the whole quantity nQ into n shipments of lot size Q to the buyer in a replenishment cycle. The buyer may not want to receive the replenishment lot immediately or in a very short time gap after reordering it because if the reordered lot arrives to the buyer's premises too early before the complete consumption of on-hand stock, then this excess stock will increase the holding cost at the buyer's side. Again, the buyer prefers to allow a stock-out situation for a short time period, but not for too long period in order to minimize the inventory cost or potential customer loss (i.e., profit loss). To meet his purposes, the buyer offers a delivery tolerance range $[t_E, t_F]$ to the vendor. If a delivery lot reaches within this interval of time no penalty cost is imposed to the vendor, but if the replenishment lot arrives beyond the aforesaid tolerance range, then a penalty cost is charged to the vendor. If shortage takes place, the pending demand of the stock-out period is completely backlogged in the next ordering cycle of the buyer. Whether the buyer wants or not, evidently it is strictly dependent on what the optimal policy should be. The correct concept would be that the buyer does not want to implement a policy that is not optimal for her/him. It may also be possible that the optimal policy may result in a "high" probably of receiving a late delivery whether any party likes or not. In other words, all depends on the cost parameters and mutual agreement.

3.2. Mathematical formulation

Let us suppose that D is the demand rate and R is the reordering point of the buyer. So, $\frac{R}{D}$ is the reorder time that is the time during which the on-hand stock is fully consumed. The replenishment lot size on each replenishment cycle of the buyer is Q. In each of the replenishment cycle, when the delivery lot arrives at the buyer's premises the buyer's inventory reaches the maximum level (S). The lead time τ to replenish the buyer's order is assumed to be stochastic in nature. Let it follows a known probability distribution within a specified range $[l, L]$. Two situations may arise depending on the length of the upper bound L of the lead time τ. Either $L > R/D$ or $L \leq R/D$. If $L \leq R/D$, then shortages never take place at the buyer's facility which contradicts our Assumption 9. Thus, the valid condition for the proposed model is $L > R/D$. Now, whatever be the length of lead time τ, there must be some time that is elapsed between the events "the order is placed" and "the order has arrived". During this period, the order is outstanding. Since only a single order is placed at reorder point and the appropriate inventory level is not allowed to be lower than the reorder level just after the arrival of a replenishment order, so, at most one order will be outstanding at any time [see Cakanyildirim et al. (2000), p. 218]. The buyer offers a delivery tolerance range beyond which a penalty cost is assessed to the vendor. Buyer's allowable delivery tolerance range is $[t_E, t_F]$, where $t_E = \frac{d_E R}{D}, 0 < d_E < 1, l < t_E < R/D$ and $t_F = \frac{d_L R}{D}, d_L > 1, R/D < t_F < L$. The inventory level of the vendor and the buyer over time is shown in Fig. 1.

In general, $0 \leq d_E \leq 1, l \leq t_E \leq R/D, d_L \geq 1, R/D \leq t_F \leq L$. In conservative systems, there may be some pessimistic buyers who will like to receive the ordered lots as early as possible. So that they do not impose any early delivery tolerance limits. In such case, $d_E = 0$ and hence $t_E = 0$. Again, there may be some buyers who do not wish to take the burden of over-stocking and so they impose the early delivery tolerance factor $d_E = 1$ which implies $t_E = R/D$ i.e., t_E is equal to the time when the on-hand inventory becomes zero. If $t_E = R/D$, we have $l < t_E = R/D < t_F < L$ by assumption. Since the lead time is stochastic, the replenishment may arrive at any time in $[l, L]$, and hence even after R/D, which means that the replenishment is not certainly early. If $t_E = l$ i.e., if the earliest delivery tolerance range is equal to the lower bound of delivery lead time τ, then the case of early delivery penalty cost will never take place. On the other hand, the buyer's holding cost increases significantly. As a result, his expected cost is also increased significantly which is not desired. Sometimes the buyer does not permit shortages; in such a situation, $d_F = 1$ and hence $t_F = R/D$. That means, if the order reaches after time $\frac{R}{D}$, i.e., if shortages take place, then a penalty cost is charged to the vendor. Also, there may be some optimistic buyers who allow shortages for a short time span. If $t_F = L$, i.e., if the latest delivery time is equal to the upper bound of the delivery tolerance range of lead time τ, then the buyer's backlogging cost becomes higher and consequently, his expected cost also becomes higher. All the cases mentioned above, do not go well with this model and they are not suitable to make the model tangible as well. The strict inequalities $0 < d_E < 1$; $l < t_E < \frac{R}{D}$; $d_L > 1$; $R/D < t_F < L$ provide an acceptable latitude for early or late delivery of ordered lots to the buyer, with the early or late delivery penalty cost being assessed to the vendor to improve delivery performance in a supply chain. An individual and integrated costs of the vendor and the buyer are discussed below.

3.2.1. Vendor's individual cost

The vendor delivers the whole contract amount nQ into n lots each of size Q (see Fig. 1 and Fig. 2). Clearly, the single replenishment cycle of the vendor is equal to the n replenishment cycles of the buyer. The cycle time of the vendor $= \frac{nQ}{D}$ and the costs relevant to the vendor's cycle are: set up cost $= \frac{C_V^v D}{nQ}$ and holding cost $= \frac{C_H^v (n-1)Q}{2}$.

Penalty cost: The vendor must face two types of nonlinear penalty costs; Type I (early delivery penalty cost) and Type II (late delivery penalty cost) if he delivers the ordered lot beyond the buyer's delivery

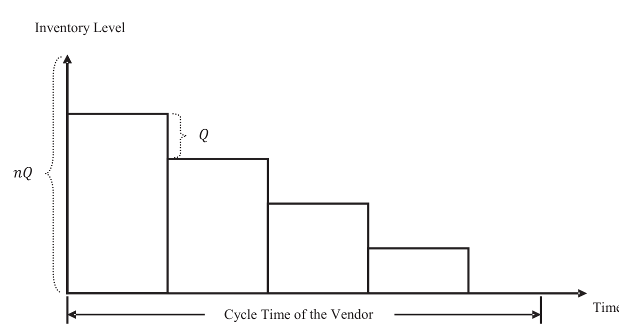

Fig. 2. Vendor's inventory level [see for reference Hossain et al. (2017)].

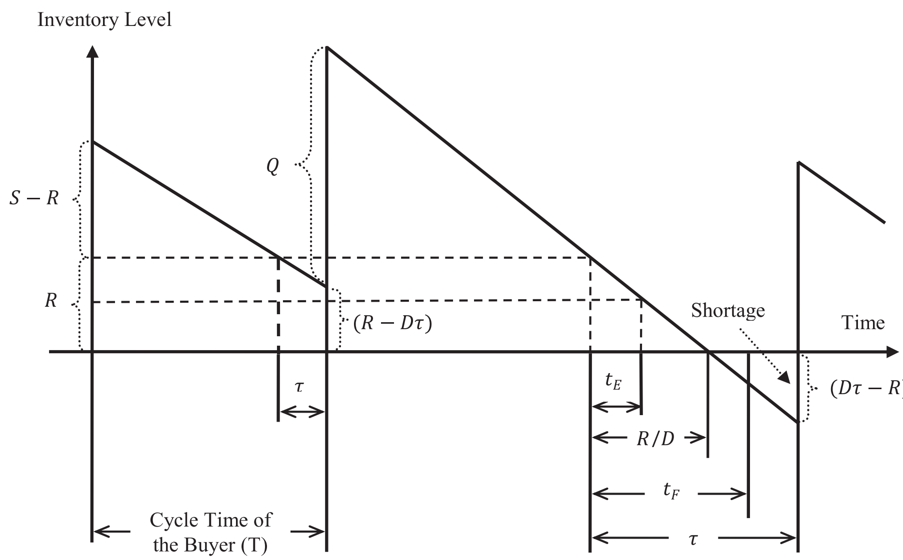

Fig. 3. Buyer's inventory level.

tolerance range $[t_E,\ t_F]$. The early and late delivery penalty costs are defined as follows.

$$PC = \begin{cases} C_E^v Q^m (t_E - \tau), \tau < t_E, 0 < m < 1 \\ C_F^v Q^m (\tau - t_F), \tau > t_F, 0 < m < 1 \end{cases}$$

where the nonlinearity factor m indicates the elasticity of the penalty costs. Here, in both cases (early and late delivery), the penalty costs are considered as a product of Q^m, where $0 < m < 1$ and a linear function of lead time τ. Therefore, the penalty costs are nonlinear in nature, where C_E^v and C_F^v are the unit penalty costs for early and late delivery, respectively.

If $m = 0$, then the penalty costs become linear functions of lead time (τ). If $m < 0$, then the penalty costs will be too small which are negligible. So, if $m > 0$, then the penalty costs become the nonlinear function of order lot size (Q) and lead time (τ). Again, if $m = 1$, then also penalty costs become nonlinear as there exist a term that is the product of Q and τ. Now, if $m \geq 1$, then the penalty costs become nonlinear but they will be significantly high that may not be possible in any supply chain collaboration. Therefore, the valid range for the parameter m is $0 < m < 1$.

These two types of penalty costs are discussed below.

(i) Type-I: *Early delivery penalty cost*

If $l \leq \tau < t_E$, i.e., if the ordered lot reaches the buyer's premises before the lower delivery tolerance period t_E, then the vendor has to face a penalty cost at a rate $\$C_E^v$/unit/year for such an early delivery. Thus, the expected penalty cost assessed to the vendor for arriving delivery lot before t_E amounts to

$$EP_E = C_E^v Q^m \int_l^{t_E} (t_E - \tau) f(\tau) d\tau.$$

(ii) Type-II: *Late delivery penalty cost*

If $t_F < \tau \leq L$, i.e., if the delivery lot reaches the buyer's facility after the upper delivery tolerance period t_F, then the vendor must pay a penalty cost at a rate $\$C_F^v$/unit/year. Thus, the expected penalty cost for such a late delivery amount to

$$EP_F = C_F^v Q^m \int_{t_F}^{L} (\tau - t_F) f(\tau) d\tau$$

Now, the expected total penalty cost of the vendor is

$$EPC = Q^m \left[C_E^v \int_l^{t_E} (t_E - \tau) f(\tau) d\tau + C_F^v \int_{t_F}^{L} (\tau - t_F) f(\tau) d\tau \right]$$

The expected cost of the vendor is

$$EAC_V(Q,n) = \frac{C_V^v D}{nQ} + \frac{C_H^v (n-1) Q}{2} + Q^m \left[C_E^v \int_l^{t_E} (t_E - \tau) f(\tau) d\tau + C_F^v \int_{t_F}^{L} (\tau - t_F) f(\tau) d\tau \right]. \tag{1}$$

3.2.2. Buyer's individual cost

The buyer receives the whole contract amount into n lots each of size Q. The order may arrive any time before the occurrence of shortages or at the time when the on-hand stock is completely absorbed or after the occurrence of shortages. If it arrives before the stock-out situation, then the maximum level of inventory will be $S = Q + (R - D\tau)$. If it reaches after the stock-out situation, then also $S = Q - (D\tau - R) = Q + R - D\tau$. Also, the delivery lot may reach within or beyond the delivery tolerance range $[t_E, t_F]$. The behavior of the buyer's inventory is shown in Fig. 3.

The cost relevant to the buyer's cycle is ordering cost $= \frac{C_0^b D}{Q}$.

The actual expected average inventory of the buyer is = the expected average inventory before and on the reordering point + the expected average inventory after the reordering point. In most of the inventory studies, researchers have ignored the expected average inventory before the reordering point. They have calculated only the expected average inventory after the reordering point which is not an appropriate way as it does not give the correct result. Therefore, in this paper, the on-hand inventory at the buyer's facility is calculated into two parts. Part-I: on-hand inventory before and on the reordering point and Part-II: on-hand inventory after the reordering point.

3.2.2.1. Part-I: on-hand inventory before and on the reordering point. The on-hand inventory per replenishment cycle before and on the reordering point is

$$I_0 = \frac{1}{2}(T-\tau).(S-R) + R(T-\tau) = \frac{1}{2D}\left[(Q-D\tau)^2 + 2R(Q-D\tau)\right].$$

Thus, the expected holding cost before and on the reordering point for D/Q cycles is

$$EH_0 = \frac{C_h^b D}{Q}\int_l^L \frac{1}{2D}\left[(Q-D\tau)^2 + 2R(Q-D\tau)\right]f(\tau)d\tau. \quad (2)$$

3.2.2.2. Part-II: on-hand inventory after the reordering point. As τ varies, the delivery lot may arrive any time after reordering it. It may arrive before or on the reordering time R/D or after it. So, there may arise any one of the two cases. (a) Case 1: $\tau \le R/D$ or (b) Case 2: $\tau > R/D$.

(a) **Case 1:** $\tau \le \frac{R}{D}$

In this case, shortages do not occur. Here, the amount of on-hand inventory depends on the length of the lead time. According to the arrival of replenishment order at buyer's premises the following two sub-cases may arise.

(i) *Sub-case 1.1:* $l < \tau \le t_E$

In this sub-case, the delivery lot reaches to buyer's premises before or at t_E of the buyer's delivery tolerance range $[t_E, t_F]$. So, the on-hand inventory per replenishment cycle during the lead time τ is $I_1 = \frac{1}{2}D\tau.\tau + \tau(R-D\tau) = \tau(R-\frac{D\tau}{2})$. The expected holding cost in this sub-case during the lead time τ for D/Q cycles is

$$EH_1 = \frac{C_h^b D}{Q}\int_l^{t_E} \tau\left(R-\frac{D\tau}{2}\right)f(\tau)d\tau. \quad (3)$$

(ii) *Sub-case 1.2:* $t_E < \tau \le R/D$

Here, the delivery lot reaches to buyer's facility after the time t_E. So, the on-hand stock per replenishment cycle during the lead time τ is $I_2 = \tau(R-\frac{D\tau}{2})$. The expected holding cost in this sub-case during the lead time τ for D/Q cycles is

$$EH_2 = \frac{C_h^b D}{Q}\int_{t_E}^{\frac{R}{D}} \tau\left(R-\frac{D\tau}{2}\right)f(\tau)d\tau. \quad (4)$$

(b) **Case 2:** $\tau > R/D$

In this case, shortages take place. The pending demand of the stock-out period is completely backlogged in the next replenishment. The on-hand stock per replenishment cycle in this case during the lead time τ is $I_3 = \frac{1}{2}\cdot\frac{R}{D}.R = \frac{R^2}{2D}$. The expected holding cost in this case during the lead time τ for D/Q cycles is

$$EH_3 = \frac{C_h^b D}{Q}\int_{\frac{R}{D}}^L \frac{R^2}{2D}f(\tau)d\tau. \quad (5)$$

The amount short per replenishment cycle is $= \frac{1}{2}\left(\tau-\frac{R}{D}\right)(D\tau-R) = \frac{1}{2D}(D\tau-R)^2$. So, the expected backlogging cost in D/Q cycles is

$$EB = \frac{C_b^b}{2Q}\int_{\frac{R}{D}}^L (D\tau-R)^2 f(\tau)d\tau. \quad (6)$$

Now, the expected cost of the buyer is the combination of ordering cost, all holding costs including the sub-cases and backlogging cost while he/she takes his/her decision independently. Thus, the expected cost of the buyer can be written as

$$\begin{aligned} EAC_B(Q,R) = {} & \frac{C_0^b D}{Q} + \frac{C_h^b D}{Q}\int_l^L \frac{1}{2D}\left[(Q-D\tau)^2 + 2R(Q-D\tau)\right]f(\tau)d\tau \\ & + \frac{C_h^b D}{Q}\int_l^{t_E}\tau\left(R-\frac{D\tau}{2}\right)f(\tau)d\tau + \frac{C_h^b D}{Q}\int_{t_E}^{\frac{R}{D}}\tau\left(R-\frac{D\tau}{2}\right)f(\tau)d\tau \\ & + \frac{C_h^b D}{Q}\int_{\frac{R}{D}}^L \frac{R^2}{2D}f(\tau)d\tau + \frac{C_b^b}{2Q}\int_{\frac{R}{D}}^L (D\tau-R)^2 f(\tau)d\tau. \end{aligned} \quad (7)$$

Theorem 1. If the delivery lot reaches beyond the delivery tolerance window $[t_E, t_F]$, then the penalty cost charged to the vendor affects the expected cost of the vendor but not the buyer.

Proof. In case of early or late delivery, the vendor must pay the penalty cost to the buyer. Let us assume that the expected total penalty cost to be $EPC_{penalty} = EPC$. Suppose the vendor's other cost is EAV_{other}. So, the vendor's expected total cost incurred after the penalty assessment is

$$EAC_{vendor} = EAC_V = EAV_{other} + EPC_{penalty} \quad (8)$$

Now, this penalty cost, $EPC_{penalty}$, is paid to the buyer whose other cost is $EAB_{other} = EAC_B$, but this penalty cost $EPC_{penalty}$, once paid by the vendor, is a gain to the buyer, meaning that buyer's cost is now reduced (because of the gain) by this amount of penalty cost. Hence, the resulting buyer's cost is $EAC_{buyer} = EAB_{other} - EPC_{penalty}$.

(i) Case I: Early delivery

If the delivery lot reaches before the lower limit of the delivery tolerance range, then the buyer has to bear an extra holding cost for keeping the lot an excess time before the delivery time window. This extra cost can be considered as a loss to the buyer i.e., EAB_{loss}. For this reason, the buyer charges a penalty cost to the vendor or his representative transportation agency which is equivalent to this extra cost, meaning $EAB_{loss} \equiv EPC_{penalty}$.

(ii) Case II: Late delivery

If the shipment is late, the buyer is suffering from loss of shortage of product/material during this late period, and the penalty cost assessed to the vendor or his representative transportation agency by the buyer is nothing but the equivalence of his market goodwill loss and opportunity lost for not selling the products during this time. This is the reason for charging the penalty cost, otherwise there is no justification for assessing such a cost. So, let this goodwill and opportunity loss by the buyer be EAB_{loss}. That is,

$$EAB_{loss} \equiv EPC_{penalty}$$

In either of the cases, this EAB_{loss} is a cost to the buyer and thus it needs to be added to his expected total cost EAC_{buyer}. Therefore, the buyer's total cost becomes

$$\begin{aligned} EAC_{buyer} &= EAB_{other} - EPC_{penalty} + EAB_{loss} \\ &= EAB_{other}, \text{ since } EPC_{penalty} = EAB_{loss} \\ &= EAC_B. \end{aligned} \tag{9}$$

Thus, it is clear from Eqs. (8) and (9) that the penalty cost charged to the vendor affects the expected cost of the vendor but not the buyer. Hence the proof.

3.2.3. The vendor-buyer integrated cost

The integrated expected cost is the sum of the vendor's and buyer's individual costs which can be written as

$$\begin{aligned} EACI &= EACI(Q,R,n) \\ &= EAC_V(Q,n) + EAC_B(Q,R) \\ &= \frac{D}{Q}\left(C_0^b + \frac{C_V^v}{n}\right) + \frac{C_H^v(n-1)Q}{2} \\ &+ Q^m\left[C_E^v\int_l^{t_E}(t_E-\tau)f(\tau)d\tau + C_F^v\int_{t_F}^L(\tau-t_F)f(\tau)d\tau\right] \\ &+ C_h^b\int_l^L\left\{\left(\frac{Q}{2}+\frac{D^2\tau^2}{2Q}-D\tau\right)+R\left(1-\frac{D\tau}{Q}\right)\right\}f(\tau)d\tau \\ &+ \frac{C_h^bD}{Q}\int_l^{\frac{R}{D}}\tau\left(R-\frac{D\tau}{2}\right)f(\tau)d\tau \\ &+ \frac{1}{2Q}\int_{\frac{R}{D}}^L\{C_h^bR^2 + C_b^b(D\tau-R)^2\}f(\tau)d\tau \\ &= Z(Q,R,n) + Q^m\left[C_E^v\int_l^{\frac{d_ER}{D}}\left(\frac{d_ER}{D}-\tau\right)f(\tau)d\tau + C_F^v\int_{\frac{d_LR}{D}}^L\left(\tau-\frac{d_LR}{D}\right)f(\tau)d\tau\right], \end{aligned} \tag{10}$$

So, the optimization problem is to minimize the integrated expected cost stated in Eq. (10) subject to the condition

$$l < t_E < \frac{R}{D} < t_F < L$$

4. The solution procedure

To optimize the above integrated expected cost of the proposed supply chain model, the following theorems are to be followed.

Theorem 2. If $L > R/D$, then the integrated expected cost function $EACI$ is

(i) strictly convex in n, for given Q and R,
(ii) strictly convex in Q and R, for given n.

Proof. **(i)** The second order partial derivative of Eq. (10) with respect to n is

$$\frac{\partial^2 EACI}{\partial n^2} = \frac{2DC_V^v}{Q^2n^3} > 0 \ \forall \ Q \text{ and } R.$$

Hence, $EACI$ is strictly convex in n, for given Q and R.

(ii) The second order partial derivatives of Eq. (10) with respect to Q and R are

$$\begin{aligned} \frac{\partial^2 EACI}{\partial Q^2} &= \frac{2D}{Q^3}\left(C_0^b + \frac{C_V^v}{n}\right) + \frac{(C_h^b + C_b^b)}{Q^3}\int_{\frac{R}{D}}^L(D\tau-R)^2f(\tau)d\tau \\ &+ m(m-1)Q^{m-2}\left\{C_E^v\int_l^{\frac{d_ER}{D}}\left(\frac{d_ER}{D}-\tau\right)f(\tau)d\tau + C_F^v\int_{\frac{d_LR}{D}}^L\left(\tau-\frac{d_LR}{D}\right)f(\tau)d\tau\right\} \\ &> 0, \forall Q, R. \end{aligned}$$

$$\frac{\partial^2 EACI}{\partial R^2} = \frac{Q^m}{D^2}\left\{C_E^v{d_E}^2f\left(\frac{d_ER}{D}\right) + C_F^v{d_L}^2f\left(\frac{d_LR}{D}\right)\right\} + \frac{(C_h^b + C_b^b)}{Q}\int_{\frac{R}{D}}^L f(\tau)d\tau > 0, \ \forall \ Q, R.$$

where

$$\begin{aligned} Z(Q,R,n) &= \frac{D}{Q}\left(C_0^b + \frac{C_V^v}{n}\right) + \frac{C_H^v(n-1)Q}{2} \\ &+ C_h^b\int_l^L\left\{\left(\frac{Q}{2}+\frac{D^2\tau^2}{2Q}-D\tau\right)+R\left(1-\frac{D\tau}{Q}\right)\right\}f(\tau)d\tau \\ &+ \frac{C_h^bD}{Q}\int_l^{\frac{R}{D}}\tau\left(R-\frac{D\tau}{2}\right)f(\tau)d\tau + \frac{1}{2Q}\int_{\frac{R}{D}}^L\{C_h^bR^2 + C_b^b(D\tau-R)^2\}f(\tau)d\tau. \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 EACI}{\partial Q\partial R} &= mQ^{m-1}\left\{\frac{C_E^vd_E}{D}\int_l^{\frac{d_ER}{D}}f(\tau)d\tau - \frac{C_F^vd_L}{D}\int_{\frac{d_LR}{D}}^Lf(\tau)d\tau\right\} \\ &- \frac{(C_h^b + C_b^b)}{Q^2}\int_{\frac{R}{D}}^L(R-D\tau)f(\tau)d\tau \end{aligned}$$

$$\text{(i)} \quad Q = \sqrt{\frac{2D\left(C_0^b + \frac{C_V^v}{n}\right) + \left(C_h^b + C_b^b\right)\int_{\frac{R}{D}}^L(D\tau-R)^2f(\tau)d\tau}{C_H^v(n-1) + C_h^b\int_l^Lf(\tau)d\tau + 2mQ^{m-1}\left\{C_E^v\int_l^{\frac{d_ER}{D}}\left(\frac{d_ER}{D}-\tau\right)f(\tau)d\tau + C_F^v\int_{\frac{d_LR}{D}}^L\left(\tau-\frac{d_LR}{D}\right)f(\tau)d\tau\right\}}} \tag{11}$$

For given n, the determinant of the Hessian matrix $H = \begin{pmatrix} \frac{\partial^2 EACI}{\partial Q^2} & \frac{\partial^2 EACI}{\partial Q \partial R} \\ \frac{\partial^2 EACI}{\partial R \partial Q} & \frac{\partial^2 EACI}{\partial R^2} \end{pmatrix}$ is

$$|H| = \left(\frac{\partial^2 EACI}{\partial Q^2}\right)\left(\frac{\partial^2 EACI}{\partial R^2}\right) - \left(\frac{\partial^2 EACI}{\partial Q \partial R}\right)^2$$

Now, to ensure the convexity of *EACI*, the Hessian matrix H has to be positive definite and it happens if $|H| > 0$ i.e., if

$$\left(\frac{\partial^2 EACI}{\partial Q^2}\right)\left(\frac{\partial^2 EACI}{\partial R^2}\right) - \left(\frac{\partial^2 EACI}{\partial Q \partial R}\right)^2 > 0.$$

Since, $|H|$ contains the stochastic variable τ and nonlinear terms, so it is difficult to prove $|H| > 0$ theoretically. Therefore, it is checked and verified numerically in numerical studies.

Hence the proof.

Theorem 3. If Theorem 2 holds good, then the optimal values of Q, R and n can be obtained from the following equations.

(ii) $$Q^{m+1}\left\{C_E^v d_E \int_l^{\frac{d_E R}{D}} f(\tau)d\tau - C_F^v d_L \int_{\frac{d_L R}{D}}^L f(\tau)d\tau\right\} + C_h^b DQ \int_l^L f(\tau)d\tau + D\left(C_h^b + C_b^b\right) \times \int_{\frac{R}{D}}^L (R - D\tau)f(\tau)d\tau = 0, \quad (12)$$

$$Q = \sqrt{\frac{2D\left(C_0^b + \frac{C_V^v}{n}\right) + \left(C_h^b + C_b^b\right)\int_{\frac{R}{D}}^L (D\tau - R)^2 f(\tau)d\tau}{C_H^v(n-1) + C_h^b \int_l^L f(\tau)d\tau + 2mQ^{m-1}\left\{C_E^v \int_l^{\frac{d_E R}{D}}\left(\frac{d_E R}{D} - \tau\right)f(\tau)d\tau + C_F^v \int_{\frac{d_L R}{D}}^L \left(\tau - \frac{d_L R}{D}\right)f(\tau)d\tau\right\}}}$$

(iii) $$n = \frac{1}{Q}\sqrt{2DC_V^v / C_H^v}, \text{where} \left\lfloor \frac{1}{Q}\sqrt{2DC_V^v / C_H^v} \right\rfloor \le n \le \left\lceil \frac{1}{Q}\sqrt{2DC_V^v / C_H^v} \right\rceil, n \ge 1. \quad (13)$$

Proof. For the stationary point (Q, R, n), equating $\frac{\partial EACI}{\partial Q} = 0$, $\frac{\partial EACI}{\partial R} = 0$ and $\frac{\partial EACI}{\partial n} = 0$.

(i) Now, equating $\frac{\partial EACI}{\partial Q} = 0$ implies

$$-\frac{1}{2Q^2}\left[2D\left(C_0^b + \frac{C_V^v}{n}\right) + \left(C_h^b + C_b^b\right)\int_{\frac{R}{D}}^L (D\tau - R)^2 f(\tau)d\tau\right] + \frac{C_H^v(n-1)}{2} + \frac{C_h^b}{2}\int_l^L f(\tau)d\tau + mQ^{m-1}\left\{C_E^v \int_l^{\frac{d_E R}{D}}\left(\frac{d_E R}{D} - \tau\right)f(\tau)d\tau + C_F^v \int_{\frac{d_L R}{D}}^L \left(\tau - \frac{d_L R}{D}\right)f(\tau)d\tau\right\} = 0$$

or,

(ii) Equating $\frac{\partial EACI}{\partial R} = 0$ implies

$$Q^m\left\{\frac{C_E^v d_E}{D}\int_l^{\frac{d_E R}{D}} f(\tau)d\tau - \frac{C_F^v d_L}{D}\int_{\frac{d_L R}{D}}^L f(\tau)d\tau\right\} + C_h^b \int_l^L f(\tau)d\tau + \frac{\left(C_h^b + C_b^b\right)}{Q}\int_{\frac{R}{D}}^L (R - D\tau)f(\tau)d\tau = 0$$

or,

$$Q^{m+1}\left\{C_E^v d_E \int_l^{\frac{d_E R}{D}} f(\tau)d\tau - C_F^v d_L \int_{\frac{d_L R}{D}}^L f(\tau)d\tau\right\} + C_h^b DQ \int_l^L f(\tau)d\tau + D\left(C_h^b + C_b^b\right)\int_{\frac{R}{D}}^L (R - D\tau)f(\tau)d\tau = 0$$

(iii) Equating $\frac{\partial EACI}{\partial n} = 0$ implies that $-\frac{DC_V^v}{Q^2 n^2} + \frac{C_H^v}{2} = 0$

or,

$$n = \frac{1}{Q}\sqrt{2DC_V^v / C_H^v}$$

As n indicates the number of deliveries from the vendor to the buyer, it must be a positive integer. Therefore, it will be within an integer range such that

$$\left\lfloor \frac{1}{Q}\sqrt{2DC_V^v / C_H^v} \right\rfloor \le n \le \left\lceil \frac{1}{Q}\sqrt{2DC_V^v / C_H^v} \right\rceil, n \ge 1$$

Hence the proof.

Theorem 4. If $m = 0$ and $C_E^v = 0$, then the present model reduces to the model described by Hossain et al. (2017) as Case 1.

Proof. If we substitute $m = 0$ and $C_E^v = 0$ in Eq. (10), then the expression for integrated expected cost turns into the expression stated by Hossain et al. (2017) in Case 1. So, the model discussed by Hossain et al. (2017) is a special case of the current model. The optimal values of delivery lot size and the number of deliveries are the same as stated by Hossain et al. (2017).

The Equations in Theorem 3 are intrinsic in Q, R and n. So, it is impossible to derive a closed-form solution for each of them. Hence, to determine the optimal result, an iterative procedure is described using a flow chart in Fig. 4. Based on the flow chart (see Fig. 4), a solution algorithm for the iteration procedure is stated below.

Solution Algorithm: Finding solutions for Q, R, and n

Step 1. Initialize $n = 1$, $EACI^* = \infty$, and the desired accuracy level ϵ.
Step 2. Evaluate Q_n from Eq. (13).
Step 3. Set $Q_{0n} = Q_n$.
Step 4. Utilize Q_{0n} to calculate the value of R_n from Eq. (12).
Step 5. Compute Q_n from Eq. (11).
Step 6. If $|Q_{0n} - Q_n| < \epsilon$, then go to *Step* 7. Else go to *Step* 3.
Step 7. Find $EACI_n$ from Eq. (10). Print Q_n, R_n, n and $EACI_n$.
Step 8. If $EACI_n < EACI^*$, then go to Step 9. Else go to *Step 10*.
Step 9. Set $n = n + 1$ and go to *Step 2*.
Step 10. Set $EACI^* = EACI_n, Q^* = Q_n, R^* = R_n, n^* = n$.
Step 11. Print Q^*, R^*, n^* and $EACI^*$.
Step 12. Stop.

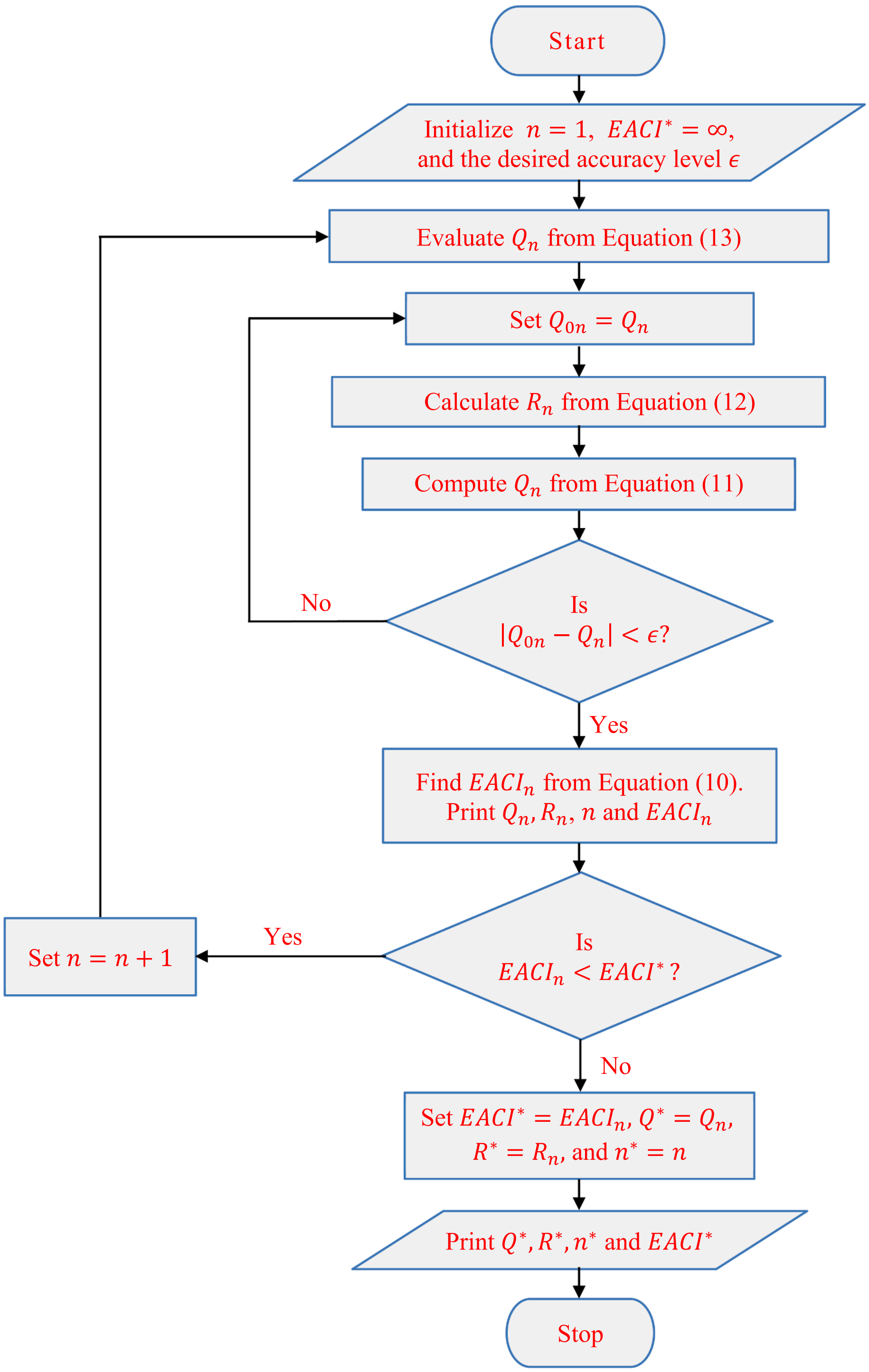

Fig. 4. Flow chart of the iterative procedure for finding the optimal solution.

Here, (Q^*, R^*, n^*) is the global optimal solution and the optimal values of the lower and upper thresholds of the delivery tolerance range, i.e., t_E^* and t_F^*, are respectively obtained from the relations $t_E^* = \frac{d_E R^*}{D}$, $0 < d_E < 1, t_E^* \in \left(l, \frac{R^*}{D}\right)$ and $t_F^* = \frac{d_L R^*}{D}, d_L > 1, t_F^* \in \left(\frac{R^*}{D}, L\right)$

In the following section, few numerical examples as experienced in industrial and/or business environments are provided to show the computations of the model in the field.

5. Numerical results

The following numerical examples are discussed to establish the general model. The parameter values are taken from the model of Hossain et al. (2017) except the new ones.

Example 1. *Uniformly distributed lead time.*

Let us consider a small-scale industry that produces toys for kids and supplies them to a retailer. The delivery lead time follows a uniform distribution. The parametric values of this system are: $D = 1000$toys/

year, $C_V^v = \$400$/setup, $C_0^b = \$25$/order, $C_H^v = \$4$/toy/year, $C_h^b = \$5$/toy/year, $C_b^b = \$30$/toy/year, $C_E^v = \$2500$/toy/year, $C_F^v = \$2190$/toy/year, $m = 0.4$, $d_E = 0.75$ (replenishment is made after passing 75% of the reorder time R/D), $d_L = 1.7$ (replenishment is done after passing 170% of the reorder time R/D) and $\tau \sim U[0, 35]$ days.

Now, the Solution Algorithm (see Section 4) or flow chart (see Fig. 4) is used to find the optimal solution. At first, $n = 1$ is substituted in Eq. (13) which yields $Q_{01} = 447$ toys/order. Then, this Q_{01} is utilized in Eq. (12) which results R_1. Now, the insertion of this R_1 in Eq. (11) yields a new value of Q_1. After several iterations for calculating Q_1 and R_1 from Eqs. (11) and (12) as shown in the flowchart, finally the local optimal solution for $n = 1$ is obtained as $Q_1 = 399$ toys/order and $R_1 = 42$ toys. Thus, the integrated expected cost which is calculated from Eq. (10) is $EACI_1 = \$2273$/year. Similarly, the local optimal solution for $n = 2$ is determined as $Q_2 = 220$ toys/order, $R_2 = 42$ toys and the corresponding integrated expected cost $EACI_2 = \$2197$/year. Since $EACI_2 < EACI_1$, so the local optimal solution for $n = 3$ is calculated and is found as $Q_3 = 155$ toys/order, $R_3 = 43$ toys and $EACI_3 = \$2208$/year. Clearly, $EACI_2 < EACI_3$, so the iteration process stops. All the local optimal solutions are recorded in Table 2.

Table 2 shows that $n = 2$ incurs lower integrated expected cost than for $n = 1$ and $n = 3$. So, the global optimal solution is found at the stationary point (Q_2, R_2) for $n = 2$. Hence, the global optimal solution is: $Q^* = 220$ toys/order, $R^* = 42$ toys, $n^* = 2$ deliveries (shipments), and $EACI^* = \$2197$/year. Here, $\left(\frac{\partial^2 EACI}{\partial Q^2}\right)_{(Q^*,R^*)} = 0.0432013 > 0$, $\left(\frac{\partial^2 EACI}{\partial R^2}\right)_{(Q^*,R^*)} = 0.787117 > 0$ and the Hessian matrix

$$H = \begin{pmatrix} \frac{\partial^2 EACI}{\partial Q^2} & \frac{\partial^2 EACI}{\partial Q \partial R} \\ \frac{\partial^2 EACI}{\partial R \partial Q} & \frac{\partial^2 EACI}{\partial R^2} \end{pmatrix}$$

is positive definite at (Q^*, R^*) since $|H|_{(Q^*,R^*)} = \left(\frac{\partial^2 EACI}{\partial Q^2}\right)_{(Q^*,R^*)} \left(\frac{\partial^2 EACI}{\partial R^2}\right)_{(Q^*,R^*)} - \left(\frac{\partial^2 EACI}{\partial Q \partial R}\right)^2_{(Q^*,R^*)} = 0.0339722 > 0$. It is also checked that the optimality conditions at the local optimal solutions are satisfied. The optimal values of the length of lower and upper delivery tolerance periods are $t_E^* = 0.0315$ year ≈ 11 days and $t_F^* = 0.0714$ year ≈ 26 days respectively. Here, $L = 35$ days which is greater than $\frac{R^*}{D} = \frac{42}{1000}$ year $\cong 15$ days. The CPU (Central Processing Unit) time needed for this calculation on *Intel Core i3*-2350M processor 2.30 GHz with 4 GB memory is 2.135 s (approximately).

Example 2. *Exponentially distributed lead time*

Suppose a company manufactures spare parts and supplies them to a retailer. The delivery lead time is exponentially distributed. The values of the parameters for this example are as follows: $D = 1000$parts/year, $C_V^v = \$50$/setup, $C_0^b = \$40$/order, $C_H^v = 1$/part/year, $C_h^b = \$4$/part/year, $C_b^b = \$6$/part/year, $C_E^v = \$2000$/part/year, $C_F^v = \$1000$/part/year, $m = 0.4$, $d_E = 0.75$ (replenishment is made after passing 75% of the reorder time R/D), and $d_L = 1.7$ (replenishment is done after passing 170% of the reorder time R/D). The lead time τ follows the probability density function $f(\tau) = 20e^{-20\tau}$ with mean $\beta = \frac{1}{20}$ year $= \frac{365}{20}$ days ≈ 18 days, $L = 35$ days and $l = 0$ day.

As stated in the Solution Algorithm, the iteration procedure begins with $n = 1$ and calculates $Q_{01} = 316$ parts/order from Eq. (13). Then by the substitution of Q_{01} in Eq. (12) determines R_1. Now Eqs. (11) and (12) are repeatedly used (see Fig. 4) to obtain the local optimal result $Q_1 = 182$ parts/order and $R_1 = 30$ parts for $n = 1$. Then, Eq. (10) yields $EACI_1 = \$1088$/year. Similarly, for $n = 2$, the local optimum values are found as $Q_2 = 143$ parts/order, $R_2 = 30$ parts and $EACI_2 = \$1015$/year. Here, $EACI_2 < EACI_1$. So, the process proceeds for $n = 3$. The local optimal solution for $n = 3$ is $Q_3 = 125$ parts/order and $R_3 = 31$ parts. Eq. (10) results $EACI_3 = \$1020$/year. Here $EACI_3$ is higher than $EACI_2$, so the iteration process ends. The local optimal results are recorded in Table 3.

It is noted from Table 3 that the minimum integrated expected cost arises for $n = 2$. So, the global optimal solution is obtained at (Q_2, R_2) for $n = 2$. Hence, the global optimal solution is: $Q^* = 143$ parts/order, $R^* = 30$ parts, $n^* = 2$ deliveries (shipments), and $EACI^* = \$1015$/year. Again, $\left(\frac{\partial^2 EACI}{\partial Q^2}\right)_{(Q^*,R^*)} = 0.0470798 > 0$, $\left(\frac{\partial^2 EACI}{\partial R^2}\right)_{(Q^*,R^*)} = 0.372987 > 0$ and the Hessian matrix H is positive definite at (Q^*, R^*) since $|H|_{(Q^*,R^*)} = 0.0175333 > 0$. Also, it is noted that the optimality conditions at the local optimal solutions for Example 2 are satisfied. The optimal values of $t_E^* = 0.0225$ year ≈ 8 days and $t_F^* = 0.051$ year ≈ 19 days. Here, $L = 35$ days which is greater than $\frac{R^*}{D} = \frac{30}{1000}$ year $\cong 11$ days. The CPU time noted for obtaining the results is 3.59 s (approximately).

Example 3. Normally distributed lead time

A steel manufacturing industry produces steel rolls with a certain gauge and supplies it to a buyer. The rolls are sold based on the weights of the rolls (i.e., the selling unit is in tonnage). The values of the system input data are: $D = 120,000$ tons/year, $C_V^v = \$1000$/setup, $C_0^b = \$560$/order, $C_H^v = \$1$/ton/year, $C_h^b = \$1.25$/ton/year, $C_b^b = \$1.5$/ton/year, $C_E^v = \$2500$/ton/year, $C_F^v = \$2400$/ton/year, $d_E = 0.75$ (replenishment is made after passing 75% of the reorder time R/D), $d_L = 1.7$ (replenishment is done after passing 170% of the reorder time R/D), $L = 35$ days, $l = 0$ day, and $m = 0.2$. The delivery lead time follows a normal distribution with $\tau \sim N[27, 12^2]$ days.

Now, with the help of the flow chart which is shown in Fig. 4, at first $Q_{01} = 15491$ tons/order is computed from Eq. (13) by putting $n = 1$. Then this Q_{01} is used in Eq. (12) and obtained R_1. Now Eqs. (11) and (12) are continuously used according to the flow chart to determine the local optimal solution for $n = 1$ i.e., $Q_1 = 29660$ tons/order and $R_1 = 2871$ tons. Then, Eq. (10) results $EACI_1 = \$10047$/year. Similarly, for $n = 2$, the local optimal values are found as $Q_2 = 13689$ tons/order, $R_2 = 5005$ tons and $EACI_2 = \$16680$/year. The iteration process terminates since $EACI_1 < EACI_2$. These local optimal outcomes are recorded in Table 4.

Since, the integrated expected cost for $n = 2$ is higher than $n = 1$ (see Table 4), so the global optimal solution is found at (Q_1, R_1) for $n = 1$. Hence, the global optimal solution is: $Q^* = 29660$ tons/order, $R^* = 2871$ tons, $n^* = 1$ delivery (shipment), and the corresponding integrated expected cost $EACI^* = \$10047$/year. Also, $\left(\frac{\partial^2 EACI}{\partial Q^2}\right)_{(Q^*,R^*)} =$

Table 2
The local optimal solutions for given values of n (Example 1).

n (shipments)	Q_n (toys)	R_n (toys)	$EACI_n$ ($/year)
1	399	42	2273
2	**220**	**42**	**2197**
3	155	43	2208

Table 3
The local optimal solutions for given values of n (Example 2).

n (shipments)	Q_n (parts)	R_n (parts)	$EACI_n$ ($/year)
1	182	30	1088
2	**143**	**30**	**1015**
3	125	31	1020

Table 4
The local optimal solutions for given values of n (Example 3).

n (shipments)	Q_n (tons)	R_n (tons)	$EACI_n$ ($/year)
1	**29,660**	**2871**	**10,047**
2	13,689	5005	16,680

Table 5
Sensitivity analysis for Example 1 when $n = 2$ and τ is uniformly distributed.

Parameters	Parameters Values	Q^* (Toys)	R^* (Toys)	$EACI^*$ ($/year)
D	500	155	21	1572
	750	189	32	1910
	{1000}	220*	42*	2197*
	1250	248	52	2452
	1500	274	63	2685
C_V^v	200	166	43	1680
	300	195	42	1957
	{400}	220*	42*	2197*
	500	243	42	2412
	600	264	42	2609
C_0^b	12.5	214	42	2139
	18.75	217	42	2168
	{25}	220*	42*	2197*
	31.25	223	42	2225
	37.5	226	42	2253
C_H^v	2	248	42	1963
	3	233	42	2083
	{4}	220*	42*	2197*
	5	209	42	2305
	6	200	42	2407
C_h^b	2.5	256	45	1908
	3.75	236	43	2059
	{5}	220*	42*	2197*
	6.25	207	40	2325
	7.5	197	39	2443
C_b^b	15	218	41	2178
	22.5	219	41	2188
	{30}	220*	42*	2197*
	37.5	221	42	2206
	45	222	43	2214
C_E^v	1250	222	46	2135
	1875	221	44	2167
	{2500}	220*	42*	2197*
	3125	219	40	2224
	3750	219	39	2250
C_F^v	1095	224	34	2153
	1642.5	222	39	2180
	{2190}	220*	42*	2197*
	2737.5	220	44	2209
	3285	219	46	2217
m	0.2	228	36	2079
	0.3	225	40	2124
	{0.4}	220*	42*	2197*
	0.5	212	43	2316
	0.6	199	44	2510

Note: {}- the base row; * - the optimal solution.

$0.000016 > 0$, $\left(\frac{\partial^2 EACI}{\partial R^2}\right)_{(Q^*,R^*)} = 0.000076 > 0$ and the Hessian matrix H is positive definite at (Q^*, R^*) since $|H|_{(Q^*,R^*)} = 1.15524 \times 10^{-9} > 0$. Also, it is found that the optimality conditions at the local optimal solutions are satisfied. The optimal values of $t_E{}^* = 0.0179$ year ≈ 7 days and $t_F{}^* = 0.0407$ year ≈ 15 days. Here, $L = 35$ days which is greater than $\frac{R^*}{D} = \frac{2871}{120000}$ year ≈ 9 days. The CPU time for calculation is found as 8.468 s (approximately).

6. Sensitivity analyses

The sensitivity analyses of the optimal solutions for Examples 1, 2 and 3 are performed by changing the values of the parameters $D, C_V^v, C_0^b, C_H^v, C_h^b, C_b^b, C_E^v, C_F^v$ and m by −50%, −25%, +25% and +50% while the other parameters value remain unaltered. The effects of such changes on the optimal values of order lot size Q^*, reorder point R^* and the integrated expected cost $EACI^*$ for Examples 1, 2, and 3 are reported in Tables 5, 6 and 7, respectively.

The scenarios observed from Tables 5–7 are as follows.

Table 6
Sensitivity analysis for Example 2 when $n = 2$ and τ is exponentially distributed.

Parameters	Parameters values	Q^* (Spare parts)	R^* (Spare parts)	$EACI^*$ ($/year)
D	500	97	16	744
	750	122	23	891
	{1000}	143*	30*	1015*
	1250	163	36	1124
	1500	181	43	1224
C_V^v	25	129	31	923
	37.5	136	30	970
	{50}	143*	30*	1015*
	62.5	149	30	1057
	75	156	30	1098
C_0^b	20	121	30	864
	30	132	30	942
	{40}	143*	30*	1015*
	50	153	30	1082
	60	163	30	1145
C_H^v	0.5	149	30	978
	0.75	146	30	997
	{1}	{143}	30*	1015*
	1.25	140	30	1033
	1.5	138	30	1050
C_h^b	2	180	36	842
	3	158	33	935
	{4}	143*	30*	1015*
	5	132	27	1084
	6	124	24	1145
C_b^b	3	141	28	1000
	4.5	142	29	1008
	{6}	143*	30*	1015*
	7.5	144	30	1022
	9	145	31	1028
C_E^v	1000	145	36	975
	1500	144	33	997
	{2000}	143*	30*	1015*
	2500	143	27	1030
	3000	142	25	1043
C_F^v	500	150	17	941
	750	146	24	984
	{1000}	143*	30*	1015*
	1250	141	34	1036
	1500	140	37	1052
m	0.2	155	19	896
	0.3	150	25	945
	{0.4}	143*	30*	1015*
	0.5	133	33	1117
	0.6	119	36	1267

Note: {}- the base row; * - the optimal solution.

- If the annual demand D increases, then the order lot size Q^* and reorder point R^* are increased. As a result, the integrated expected cost $EACI^*$ for Examples 1, 2, and 3 are also increased (see Tables 5–7).
- With an increase in the set-up cost C_V^v, the order lot size Q^* and the integrated expected cost $EACI^*$ increase but the reorder point R^* decreases for Example 1, 2, and 3 (see Tables 5–7).
- While buyer's ordering cost C_0^b increases, the values of Q^* and $EACI^*$ increase significantly for Example 1, 2, and 3. The value of R^* remains unaltered for Example 1 and 2 (see Tables 5 and 6) but decreases for Example 3 with an increase in C_0^b (see Table 7).
- When the vendor's holding cost C_H^v increases, the value of Q^* decreases, but $EACI^*$ increases while R^* remains unchanged for Example 1 and 2 (see Tables 5 and 6) whereas in Example 3, Q^*, R^* and $EACI^*$ remain unaltered (see Table 7).
- The values of Q^* and R^* are decreased and $EACI^*$ is increased with an increase in the value of buyer's holding cost C_h^b for Example 1, 2, and 3 (see Tables 5–7).

Table 7
Sensitivity analysis for Example 3 when $n = 1$ and τ is normally distributed.

Parameters	Parameters values	Q^* (Tons)	R^* (Tons)	$EACI^*$ ($/year)
D	60,000	20,157	1840	7779
	90,000	25,193	2319	9065
	{120000}	29660*	2871*	10047*
	150,000	33,714	3552	10,850
	180,000	37,446	4350	11,532
C_V^v	500	24,789	3445	7843
	750	27,338	3144	8994
	{1000}	29660*	2871*	10047*
	1250	31,812	2609	11,023
	1500	33,832	2346	11,937
C_0^b	280	27,046	3178	8862
	420	28,384	3021	9468
	{560}	29660*	2871*	10047*
	700	30,883	2724	10,602
	840	32,061	2578	11,136
C_H^v	0.5	29,660	2871	10,047
	0.75	29,660	2871	10,047
	{1}	29660*	2871*	10047*
	1.25	29,660	2871	10,047
	1.5	29,660	2871	10,047
C_h^b	0.625	40,463	4655	7898
	0.9375	33,565	3696	9166
	{1.25}	29660*	2871*	10047*
	1.5625	27,187	2105	10,681
	1.8750	25,655	1208	11,144
C_b^b	0.75	NFS	NFS	NFS
	1.125	29,769	2140	9909
	{1.5}	29660*	2871*	10047*
	1.875	29,633	3339	10,161
	2.25	29,624	3699	10,260
C_E^v	1250	29,754	2688	10,068
	1875	29,706	2780	10,058
	{2500}	29660*	2871*	10047*
	3125	29,616	2961	10,036
	3750	29,573	3051	10,024
C_F^v	1200	NFS	NFS	NFS
	1800	30,111	2174	9954
	{2400}	29660*	2871*	10047*
	3000	29,389	3338	10,117
	3600	29,204	3687	10,173
m	0.1	NFS	NFS	NFS
	0.15	30,914	1107	9883
	{0.2}	29660*	2871*	10047*
	0.25	29,048	4043	10,156
	0.3	28,769	4967	10,169

Note: {}- the base row; * - the optimal solution.

- An increase in C_b^b increases the values of Q^*, R^* and $EACI^*$ for Examples 1 and 2 (see Tables 5 and 6) while for Example 3, the value of Q^* decreases but R^* and $EACI^*$ are increased (see Table 7).
- If the early delivery penalty cost C_E^v increases, then for Example 1 and 2 (see Tables 5 and 6), Q^* and R^* decrease but $EACI^*$ increases whereas for Example 3, the values of Q^* and $EACI^*$ decrease but R^* increases (see Table 7).
- In Example 1, 2, and 3, the reorder point R^* and the integrated expected cost $EACI^*$ are increased but the order lot size Q^* is decreased when the late delivery penalty cost C_F^v is increased (see Tables 5–7).
- The values of Q^*, R^* and $EACI^*$ are highly sensitive to m. When m increases, the value of Q^* decreases but R^* and $EACI^*$ increase significantly for Example 1, 2, and 3 (see Tables 5–7).

7. Conclusions

This study presents a viable cooperative agreement between two parties in a supply chain network for a single product. It determines a delivery policy under some conditions which are experienced in many industrial situations where the demand is constant and the delivery lead is stochastic in character. The system allows shortages that are completely backlogged. Moreover, two types of nonlinear penalty costs have been incorporated: *early delivery penalty cost* and *late delivery penalty cost*. The integrated expected cost of the entire supply chain is minimized to determine the optimal values of order quantity, reorder point, number of deliveries and the optimum integrated expected cost. Numerical examples are provided for three types of distribution of lead time: uniform, exponential and normal. Because of the generalization of the lead time distribution, it is also applicable for other types of distributions. The following conclusion may be drawn from this study.

- Introduction of nonlinear penalty costs for early and late delivery increase the supply process reliability.
- The concept of early delivery penalty cost makes a supply chain system more versatile.
- An introduction of the delivery tolerance range increases the tangibility of the supply chain.
- The distinct values of d_E and d_L make the model more practical and applicable.
- The numerical study shows that this model is useful for small-scale as well as large-scale industries.

Although the system cost of the proposed model is higher than Hossain et al. (2017) model, yet it is more effective to build a tangible coordinated supply chain between the vendor and the buyer. The nonlinear penalty costs for early and late delivery may influence the individual strategies of both the vendor and the buyer. The vendor will be compelled to be more cautious regarding the delivery timing since he alone must pay the penalty and the buyer will be able to make a balance between the holding and backlogging costs for minimizing his system cost. Consequently, the integrated expected cost will be reduced, and both of the parties will be benefited. The idea of this study is expected to improve the coordination and delivery performance in a supply chain system. The model is applicable in production houses and retail businesses where the profit margin is dependent on both: the replenishment cycle and the lead time.

The present work can be extended for a multi-stage supply chain network as well. Consideration of a single-vendor, multi-buyer or a multi-vendor, multi-buyer supply chain system including constant or different types of demand pattern may also be a further extension of this study.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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