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Phase transition in decaying black holes

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We had earlier derived the most general criteria for thermal stability of a quantum black hole, with arbitrary number of parameters, in any dimensional spacetime. These conditions appeared in form of a series of inequalities connecting second order derivatives of black hole mass with respect to its parameters. Some black holes like asymptotically flat rotating charged black holes do not satisfy all the stability criteria simultaneously, but do satisfy some of them in certain region of parameter space. They are known as "Quasi Stable" black holes. In this paper, we will show that quasi stable black holes although ultimately decay under Hawking radiation undergo phase transitions. These phase transitions are different from phase transition in ADS Schwarzschild black hole. These are marked by sign changes in certain physical quantities apart from specific heat of the black hole. We will also discuss the changes in the nature of fluctuations of the parameters of these quasi stable black holes with different phases.

Keywords: Quasi stable black hole; phase transition in black hole; non Hawking-Page phase transition; quantum gravity.

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1. Introduction

Einstein predicted, through his classical field equations of general theory of relativity, that black holes accrete everything surrounding them.[1,2] Thus a black hole will grow in size for ever. In his theory, Einstein entirely treated spacetime classically. It was Hawking who first tried to invoke quantum mechanics, through his semi classical theory,[3] to study interaction of black holes with matters surrounding them. He showed that black holes can radiate and hence decay. Thus a black hole can both accrete and radiate simultaneously.

But Hawking treated only matters quantum mechanically, not spacetime. In his theory, black holes were still classical. Thus in his semi classical theory, black holes and matters were not in equal footing. We had focused on this issue in our earlier works.[4,5] Of course, quantization of gravity has not been fully proven yet. But, at least, we know what kind of form it should have in certain situations.[4] These are enough for us to construct grand canonical partition function of a generic black hole, assuming it to be in connection with a heat bath, here the heat bath is rest of the universe. We derived, from the conditions of convergence of grand canonical partition function, the stability criteria of a generic black hole with any number of parameters in arbitrary dimensional spacetime.[5] They came in form of a series of inequalities connecting second order derivatives of black hole mass with respect to its parameters. We found, from these criteria, that ADS black holes are stable under Hawking radiation i.e. accretion dominates over radiation for certain range of its parameters.[4]

We also have noticed that asymptotically flat rotating charged black holes do not satisfy all the stability criteria simultaneously. But they satisfy some of the stability criteria within certain region of spacetime.[4,5] Thus these black holes, although decay under Hawking radiation, are different from unstable black holes, like asymptotically flat schwarzschild black hole. These black holes are categorized as "Quasi Stable" black holes.

We had calculated the fluctuations for various parameters of a stable black hole and they turned out to be very small.[6] This is very much expected for a stable system. In fact, these tiny fluctuations are the indications of the stability for a black hole. We also had calculated fluctuations of various parameters for quasi stable black holes and it turned out that some of the fluctuations are tiny,[7] like stable black holes, in certain region of parameter space. This is due to the fact that quasi stable black holes do satisfy some of the stability criteria. In fact, we had also shown that due to these facts quasi stable black holes slow down their decay rate in certain regime of their parameter space.[7]

Like ordinary thermodynamic system, black holes also have different phases. Stable black holes have stable phases within its region of stability, e.g. ADS black holes. Similarly, unstable black holes, like asymptotically flat schwarzschild black hole, have unstable phases. During their decay, unstable black holes never change its phase. Similarly stable black holes maintain equilibrium with their surrounding and stay on stable phase, preventing hawking decay. But situation is entirely different for quasi stable black holes. So far, we know that sign of specific heat distinguishes stable phase from unstable phase. In this paper, we will show that quasi stable black holes also have various different phases. Quasi stable black holes undergo phase transition among these phases during their decay process. We will in fact also show that nature of fluctuations change from one phase to another phase.

This paper is organized as follows: A detailed discussion on quasi stable black holes and their phase structure are done in Sec. 2. We also have discussed the possibility of phase transition for quasi stable black holes. In this section, we have to recapitulate some of our earlier works for sake of completeness. In the next section, we have considered two examples of quasi stable black holes and have discussed their phase transitions in details. We have also discussed on varying nature of fluctuations from one phase to another. In the last section, we summarized our work with possible future outlook.

2. Quasi Stable Black Holes and Their Phase Structure

Any charged rotating black hole possesses discrete values of charge, area and angular momentum. This is expected in any quantum theory of gravity, e.g. Loop Quantum Gravity supports this.[8] Now we can consider a black hole to be immersed in a heat bath, with which it can exchange energy, charge (Q) and angular momentum (J). Thus, we can write down the grand canonical partition function (Z_G) as summation over possible eigenstates with appropriate weightage.[9] We can convert this summation, with the help of Poisson's resummation formula,[10] into integration and determine the criteria for thermal stability. A charged rotating black hole, in thermal equilibrium, is represented by the saddle point ($\bar{A}$, $\bar{Q}$, $\bar{J}$). $\bar{A}$ denotes horizon area (A) at equilibrium and so on. It has been shown earlier[4] that this partition function turned out to be integration over the space of fluctuations $a = (A - \bar{A})$, $q = (Q - \bar{Q})$, $j = (J - \bar{J})$ around the saddle point and is given as[4]

$$Z_G \approx \int da\, dq\, dj \exp\left(-\frac{\beta}{2}\left[\left(M_{AA} - \frac{S_{AA}}{\beta}\right)a^2 + (M_{QQ})q^2 + (M_{JJ})j^2 + (2M_{AQ})aq + (2M_{AJ})aj + (2M_{QJ})qj\right]\right). \tag{1}$$

The only assumption we have made is that the mass of the black hole (M) is a function of its charge, area and angular momentum. Here, $M_{AA} = \frac{\partial^2 M}{\partial A^2}$, $M_{AQ} = \frac{\partial^2 M}{\partial A \partial Q}$ etc. and these are evaluated at the saddle point.

The convexity of the above integral leads to the criteria for thermal stability of the black hole[4] and are given as

$$(\beta M_{AA} - S_{AA}) > 0, \quad M_{QQ} > 0, \quad M_{JJ} > 0,$$
$$(M_{QQ}M_{JJ} - (M_{JQ})^2) > 0, \quad (M_{JJ}(\beta M_{AA} - S_{AA}) - \beta(M_{AJ})^2) > 0,$$
$$(M_{QQ}(\beta M_{AA} - S_{AA}) - \beta(M_{AQ})^2) > 0, \quad |H| > 0$$

where $|H|$ is the determinant of the Hessian matrix (H) and this matrix is given as

$$H = \begin{pmatrix} \beta M_{AA} - S_{AA} & \beta M_{AQ} & \beta M_{AJ} \\ \beta M_{AQ} & \beta M_{QQ} & \beta M_{JQ} \\ \beta M_{AJ} & \beta M_{JQ} & \beta M_{JJ} \end{pmatrix}.$$

We have realistically assumed that (inverse) temperature β is positive.

We get the above seven stability criteria for a black hole with two parameters (excluding horizon area), here namely charge and angular momentum. In general for black holes with "n" parameters (excluding horizon area), there will be $(2^{n+1} - 1)$ stability criteria.[5] But all these conditions are not independent, actually only $(n+1)$ number of conditions are independent. There will be exactly m number of conditions for thermodynamic equilibrium between two systems in connection whose entropy depends on m parameters. Thus for charged rotating black holes m equals to $(n+1)$. When a black hole, e.g. ADS Kerr Newman black hole,[4] satisfies all the stability criteria together within certain regime of parameter space, then the black hole is stable within that regime. We have already shown that asymptotically flat black holes with charge and rotation are quasi stable i.e. they satisfy some of the stability criteria, but not all simultaneously.

Thus we see that stability of a black hole is determined by the signs of the functions, appeared in the stability criteria. There will be $(n+1)$ number of fluctuations corresponding to $(n+1)$ number of parameters, including the area of the black hole. These fluctuations are individually related, to be shown later, with some physical quantities of the black hole. Signs of each of these physical quantities designate one distinguished phase. Thus a quasi stable black hole with "n" parameters (excluding horizon area) can at most have 2^{n+1} number of phases. Any of these physical quantity can have same sign in different isolated islands of parameter space. Thus that black hole can be in same phase in different positions of its parameter space. It can be so happened that a decaying black hole can be again in the same phase in which it was also in earlier, at its younger age. Thus quasi stable black holes go through phase transitions. Same phase transitions may reoccur many times. These interesting phase transitions can never occur either in stable or in unstable black holes. The nature of spacetime determines these phases for a quasi stable black hole.

Kerr-sen (KS) and asymptotically flat Kerr-Newman (KN) black holes are well studied quasi stable black holes.[4,6] We have already shown that M_{QQ}, M_{JJ}, $(M_{QQ}M_{JJ} - (M_{JQ})^2)$ are always positive for them, while $|H|$ is always negative. We have already shown[6] that the fixed signs of the last two quantities made the fluctuation of area always large. Hence physical quantity related to area fluctuation, e.g. it was specific heat for Schwarzschild black hole, will also have fixed sign. Thus at most four phases can be realized for both of these two black holes. In fact this is the usual scenario with the number of phases for any quasi stable black hole.

We have shown in our earlier works[6,7] that finite fluctuations of various parameters for both stable and quasi stable black holes are related to the stability criteria. We will show here that those fluctuations are related to certain physically measurable quantities of the black hole. In fact these quantities change sign during phase transitions. This idea is in fact generalization, in case of quasi stable black holes, of Hawking's old idea in the context of asymptotically flat schwarzschild black hole (AFSBH).[11] Hawking showed that AFSBH is unstable as specific heat is negative. It is in fact the reason behind the negativity of the quantity $\frac{\partial T}{\partial A}$.[4] In terms of fluctuation, this is the reason for divergence of ΔA^2 i.e. ΔA^2 becomes very large for AFSBH.[6] AFSBH has horizon area as its only parameter. But quasi stable black holes have multiple parameters, apart from horizon area. Thus it is natural to expect that fluctuations of other parameters do have similar relationship with some physical quantities. We will show that this is in fact the case.

We will now use the summation formalism of partition function to build up various physical quantities in connection with quasi stable black holes.

In this formalism, grand canonical partition function is given as[4]

$$Z_G = \sum_r \exp(-\beta(E_r - \Phi Q_r - \Omega J_r));$$

here summation is taken over eigenstates. Φ and Ω, respectively denotes electric potential and rotational speed of the isolated horizon.

Define $\bar{\Phi} \equiv \beta\Phi$, $\bar{\Omega} \equiv \beta\Omega$, where $\bar{\Phi}$ and $\bar{\Omega}$, respectively determines the electrical and rotational equilibrium between two connected systems. This is shown in details in the Appendix.

Thus partition function can be rewritten as

$$Z_G = \sum_r \exp(-\beta E_r + \bar{\Phi} Q_r + \bar{\Omega} J_r),$$

and the average value of angular momentum defined as

$$\bar{J} \equiv \frac{\sum_r J_r \cdot \exp(-\beta E_r + \bar{\Phi} Q_r + \bar{\Omega} J_r)}{Z_G} = \partial(\ln(Z_G))/\partial\bar{\Omega}.$$

Similarly we can calculate $\bar{J^2}$ and is given as

$$\bar{J^2} \equiv \frac{\sum_r J_r^2 \cdot \exp(-\beta E_r + \bar{\Phi} Q_r + \bar{\Omega} J_r)}{Z_G}.$$

We can calculate fluctuation of angular momentum and this turns out to be

$$\Delta(J)^2 \equiv \frac{\sum_r (J_r - \bar{J})^2 \cdot \exp(-\beta E_r + \bar{\Phi} Q_r + \bar{\Omega} J_r)}{Z_G}$$
$$= \bar{J^2} - (\bar{J})^2 = \partial^2(\ln(Z_G))/\partial\bar{\Omega}^2.$$

It is to be mentioned that the above calculations are possible only when fluctuation of angular momentum is converging. Moreover, the above partial derivatives are taken when β and $\bar{\Phi}$ are assumed to be constant. Similarly β and $\bar{\Omega}$ are assumed to be constant when partial derivatives are taken with respect to $\bar{\Phi}$ and so on.

We can now define rotational inertia of the black hole (S_J) as

$$S_J \equiv \beta \cdot \partial\bar{J}/\partial\bar{\Omega}$$

which is equivalent to $\beta \cdot \Delta(J)^2$.

One important issue to be mentioned here is as follows. All the quantities β, $\bar{\Phi}$ and $\bar{\Omega}$ can be expressed in terms of independent variables $\bar{A}$, $\bar{Q}$ and $\bar{J}$. Thus inversely $\bar{A}$, $\bar{Q}$ and $\bar{J}$ are also expressible in terms of β, $\bar{\Phi}$ and $\bar{\Omega}$. Hence the above partial derivative can be taken with respect to $\bar{\Omega}$, treating β, $\bar{\Phi}$ as constant and so on. Thus S_J can be calculated independently, so also $\beta \cdot \Delta(J)^2$. They turn out to be equal only if $\Delta(J)^2$ does not blow up. In fact $\Delta(J)^2$, for certain quasi stable black holes in some certain region of parameter space, approaches to become zero and then suddenly blows up. This marks the phase transition. But S_J vanishes at the point of phase transition and changes sign afterwards. It does not blow up there and starts to disrespect the above equality.

We can define the electric capacitance of a black hole as

$$S_Q \equiv \beta \cdot \partial\bar{Q}/\partial\bar{\Phi}$$

which equals to $\beta \cdot \Delta(Q)^2$.

The above said fact is also true for charge fluctuation, i.e. $\Delta(Q)^2$ approaches to become zero and then suddenly blows up. This marks the electrical phase transition. But S_Q vanishes at the point of phase transition and changes sign afterwards. It does not blow up there and starts to disrespect the above equality. Thus we find that sign changes of rotational inertia and electric capacitance individually mark two different phase transitions.

We have already seen that[4] determinant of Hessian ($|H|$) for quasi stable black holes are negative, e.g. KS and asymptotically flat KN black holes. We know[7] how to calculate fluctuations in case of quasi stable black holes. Therefore, we can conclude that

(1) $\Delta(Q)^2$ is finite only if $((\beta M_{AA} - S_{AA})M_{JJ} - \beta(M_{JA})^2)$ is negative.[a]
(2) $\Delta(J)^2$ is finite only if $(M_{QQ}(\beta M_{AA} - S_{AA}) - \beta(M_{AQ})^2)$ is negative.

Of course it is to be noted that $|H|$ is always negative.

Thus $((\beta M_{AA} - S_{AA})M_{JJ} - \beta(M_{JA})^2)$ and $(M_{QQ}(\beta M_{AA} - S_{AA}) - \beta(M_{AQ})^2)$ equal to zero are the conditions of phase transitions. These make respectively electric capacitance and rotational inertia vanishing at the point of phase transition.

3. Examples of Quasi Stable Black Holes and Their Phase Transitions

In this section, we will consider two examples of quasi stable charged rotating black holes. We will consider their phase transitions in details.

3.1. *Kerr-Sen black hole*

The mass (M) of this black hole depends on its parameters as[12]

$$M^2 = \frac{A}{16\pi} + \frac{Q^2}{2} + \frac{4\pi J^2}{A}.$$

The parameter space is restricted by the inequality $\frac{J}{A} < \frac{1}{8\pi}$ as temperature ($\propto M_A$) of a non extremal black hole is always positive. Now the quantity

[a] $\Delta(Q)^2$ measures the fluctuation of electric charge from its equilibrium value. It is mathematically expressed as[6,7] $\Delta(Q)^2 = \frac{\int da\, dq\, dj\, q^2 f(a,q,j)}{\int da\, dq\, dj\, f(a,q,j)}$; where $f(a,q,j) = \exp\left(-\frac{\beta}{2}\left[\left(M_{AA} - \frac{S_{AA}}{\beta}\right)a^2 + (M_{QQ})q^2 + (M_{JJ})j^2 + (2M_{AQ})aq + (2M_{AJ})aj + (2M_{QJ})qj\right]\right)$. Similarly, $\Delta(J)^2$ is defined.

$(M_{QQ}(\beta M_{AA}-S_{AA})-\beta(M_{AQ})^2)$ is negative when $\frac{J}{A}<\frac{0.4}{8\pi}$. Thus the curve $\frac{J}{A}=\frac{0.4}{8\pi}$ in parameter space is the boundary that separates stable rotational phase from unstable rotational phase. Hence rotational inertia vanishes on each points of the curve. But $((\beta M_{AA}-S_{AA})M_{JJ}-\beta(M_{JA})^2)$ is always negative and consequently electric capacitance is always positive. This indicates the electrical equilibrium of the black hole with the surrounding. Hence $\frac{J}{A}<\frac{0.4}{8\pi}$ is the region where both $\Delta(J)^2$ and $\Delta(Q)^2$ are finite for KS black hole. In this region of parameter space for KS black Hole, incoming and outgoing quanta of angular momentum and charge maintain a perfect equilibrium. But the equilibrium condition between incoming and outgoing quanta of angular momentum is lost in the region $\frac{0.4}{8\pi}<\frac{J}{A}<\frac{1}{8\pi}$, although the same for quanta of charge is still intact.

This KS black holes ultimately decay as $\Delta(A)^2$ is always negative. Suppose the angular momentum (J) is such that $\frac{J}{A}<\frac{0.4}{8\pi}$ and hence J will almost remain unchanged as $\Delta(J)^2$ is extremely tiny in this region. But area (A) will decrease and hence the ratio $\frac{J}{A}$ will increase and becomes greater than $\frac{0.4}{8\pi}$. Once this ratio crosses that value, J starts to fluctuate rapidly. But this ratio can not be grater than $\frac{1}{8\pi}$ with decreasing area (A). Thus J will ultimately reduce and hence the ratio $\frac{J}{A}$ becomes lesser than $\frac{0.4}{8\pi}$. This process will go on. This means that KS black hole tries to minimize the fluctuation for angular momentum. Thus we see that whenever the ratio $\frac{J}{A}$ tries to exceed $\frac{1}{8\pi}$, it reduces automatically to maintain certain bound. Hence its rotational inertia becomes negative and then again becomes positive. This in fact goes on. Thus a particular phase transition happens in various places of parameter space. This is possible only for quasi stable black holes. Phase transition labeled by change in sign of electric capacitance does not occur in case of KS black holes as charge fluctuation never blows up for it.

3.2. *Kerr-Newman black hole*

The mass (M) of this black hole depends on its parameters as[13]

$$M^2=\frac{A}{16\pi}+\frac{\pi}{A}(4J^2+Q^4)+\frac{Q^2}{2}.$$

The parameter space is restricted by the inequality $(4J^2+Q^4)<\frac{A^2}{16\pi^2}$ as temperature of a non extremal black hole is always positive. Unlike the case of KS black hole, here $\frac{Q^2}{A}$ ratio is also bounded and its maximum value is $\frac{1}{4\pi}$, double of that for $\frac{J}{A}$.

Define, $x\equiv\pi J/A$ and $y\equiv\pi Q^2/A$. We introduce these two dimensionless quantities to reduce complexity in writing some expressions. Now on calculation it turns out that $(M_{QQ}(\beta M_{AA}-S_{AA})-\beta(M_{AQ})^2)$ is proportional to F_{AQ}, where

$$\begin{aligned}F_{AQ}\equiv{}&\frac{9y^3}{32}-34x^2y^3+9x^2y^2+9x^2y-\frac{7y^4}{8}-13y^5-\frac{3y^2}{64}\\&-\frac{3y}{512}+\frac{x^2}{16}+6x^4+72x^4y-\frac{1}{2048}.\end{aligned}$$

Thus the two-dimensional surface $F_{AQ} = 0$ in three-dimensional parameter space is the boundary that separates stable rotational phase from unstable rotational phase. Hence this surface is surface of null rotational inertia. So crossing of the black hole through this surface indicates the occurrence rotational phase transition.

In Fig. 1, the shaded region is the projected region of parameter space where angular momentum fluctuates very tiny, i.e. $F_{AQ} < 0$. Thus rotational inertia is positive in this region. This figure manifestly indicates that higher values of $\frac{J}{A}$ make the fluctuation of angular momentum large. Hence even if $\frac{J}{A}$ is large at the beginning, J will reduce due to its large fluctuation as area of the decaying black hole reduces. $\frac{J}{A}$ ratio will come into the region as shown in Fig. 1, hence J would not change much. But area (A) will decrease continuously and as a result $\frac{J}{A}$ ratio will again become large enough such that J starts to fluctuate appreciably once again. This switching of $\frac{J}{A}$ ratio from larger to smaller value and vice versa will keep going. This is same as the case of KS black hole. Thus here also phase transition labeled by vanishing of rotational inertia occurs and it can occur anywhere in $F_{AQ} = 0$ surface.

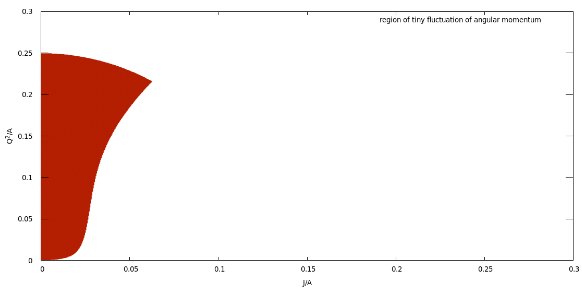

Fig. 1. Pictorial representation of region of tiny fluctuation of angular momentum.

Similarly $((\beta M_{AA} - S_{AA})M_{JJ} - \beta(M_{JA})^2)$ can be expressed in terms of x and y. It turns out on calculation that this is proportional to F_{AJ}, where

$$F_{AJ} \equiv \frac{5y^2}{8} + 7y^4 + 4y^3 - 16x^2 - \frac{1}{256}.$$

Thus the two-dimensional surface $F_{AJ} = 0$ in three-dimensional parameter space is the boundary that separates stable electrical phase from unstable electrical phase. Hence this surface is surface of null electric capacitance. So crossing of the black hole through this surface indicates the occurrence electrical phase transition.

In Fig. 2, the shaded region is the projected region of parameter space where electric charge fluctuates very tiny i.e. $F_{AJ} < 0$. Thus electric capacitance is positive in this region. It manifestly indicates that higher values of $\frac{Q^2}{A}$ make the fluctuation of charge tiny only if the ratio $\frac{J}{A}$ is high as well. But we have just seen that $\frac{J}{A}$ ratio cannot always be high. In fact the ratio $\frac{Q^2}{A}$ will oscillate between higher and lower

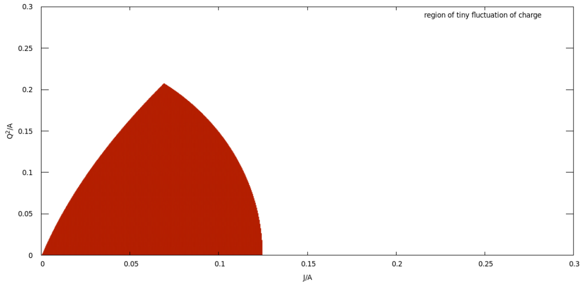

Fig. 2. Pictorial representation of region of tiny fluctuation of charge.

values, exactly as the reason $\frac{J}{A}$ ratio does the same. This indicates the occurrence of phase transition labeled by change in sign of electric capacitance. It can occur anywhere in $F_{AJ} = 0$ surface. This particular type of phase transition was absent for KS black hole. This difference is due to the fact that positivity of temperature restricts both charge and angular momentum for KN black hole, but in case of KS black holes, it is angular momentum only.

4. Discussion

Quasi stable rotating charged black holes, like unstable black holes, ultimately decay under Hawking radiation. But charge and angular momentum do not fluctuate appreciably for these black holes in certain region of their parameter spaces. This is somewhat similar to stable black holes. But the unique property of these quasi stable black holes are multiple phase transitions, although they are decaying. AFSBH is unstable as specific heat diverges. ADS Schwarzschild black holes become unstable when specific heat becomes negative. The sign change is specific heat indicates the phase transition. But ADS Schwarzschild black hole has no parameter except its horizon area. Thus that was the only possible phase transition. But a quasi stable black hole can have too many phases. How would those phase transitions look like? This is very natural question to ask. In this paper we have answered this question explicitly with examples. In this paper we also have tried to give a geometric interpretation of these phase transitions. These are unique features for phase transitions in quasi stable black holes. This is the novelty of this paper.

There exist models like Ruppeiner geometry, non-extended phase space analysis to study phase transitions of black holes. The starting point of Ruppeiner geometry is to express entropy of a black hole in terms of its mass, charge and angular momentum.[14] One then considers the change in entropy upto second order in terms of changes in mass, charge and angular momentum. This change is then expressed in form of a Riemannian line element and thermodynamic scalar is calculated from there. Divergence in this scalar indicates the microscopic disorder.[15] In this formalism, choice of thermodynamic variables are ambiguous. But this is not the case with our formalism, where the starting point is the construction of grand canonical partition function. Here we have to take horizon area, electric charge and angular momentum as thermodynamic variables as mass of the black hole solely depends on these parameters.[4] We found that the entire partition function of the black hole turned out to be same for the boundary states only. This is possible due to thermal holography i.e. complete annihilation of the bulk states of the black hole.[4] Bulk states respect symmetries like local Lorentz invariance, Gauss's constraint of electrodynamics (local gauge invariance) and as a consequence their respective generators angular momentum, electric charge appear in form of thermodynamic variables in expression of grand canonical partition function. Thus we see that our formalism is structurally quite different from that of Ruppeiner's geometrical formalism to study phase transition. In fact our formalism is also applicable for non conventional black holes as well.[5] Hence independent identity and applicability of our formalism to study black hole thermodynamics is still intact even after consideration of Ruppeiner's geometrical formalism to study phase transition of black holes. In this paper, we find that same phase transition occurs multiple times for quasi stable black holes during their decay, having similarity with reentrant phase transition in case of AdS Kerr-Newman black hole.[16] This feature in fact indicates the self sufficiency and independency of our formalism.

On the other hand one choose square of electric charge, not charge itself, as thermodynamic variable in non-extended phase space formalism.[17] But we have already mentioned that choice of electric charge as thermodynamic variable is automatic in our formalism due to Gauss's constraint of electrodynamics of bulk spacetime.[4] The form of electric capacitance, considered in this paper, in connection with electric charge would definitely be changed if we consider square of electric charge as thermodynamic variable. But the overall conclusion like thermal instability of Kerr-Newman black hole should remain unaltered. Actually electric capacitance, connected with thermodynamic variable electric charge, is very natural physical quantity to consider even for thermodynamics of ordinary electric system. Thus Gauss's constraint of electrodynamics of bulk spacetime guarantees the non violation of this natural outcome.

In fact this paper can have imprints on other branch of physics as well. We have seen here that rotating charged quasi stable black holes would end up in non-rotating black holes by virtue of multiple numbers of phase transitions. Close to the end state, they would become a tiny ball with a minimum area according to theory like LQG. Thus they can form component of dark matter as well.[18] In this sense, our analysis may have some impacts on dark matter physics.

We have also noticed that fluctuations for charge and angular momentum of quasi stable rotating charged black holes have some similarities with that of stable ADS Kerr-Newman black holes. Now, the ADS/CFT correspondence tells that asymptotically ADS black hole is dual to a strongly coupled gauge theory at finite temperature.[19–22] It is possible to analyze the strongly correlated condensed matter physics using ADS/CFT correspondence. Thus this paper may have some impacts on condensed matter physics as well.

Appendix

Consider two systems A and B are in contact and they together form an isolated system. Suppose they are allowed to share energy only, but total energy $(E_A + E_B)$ is conserved. Thus total entropy (S) of this isolated system $(A+B)$ is $(S_A(E_A) + S_B(E_B))$. In thermal equilibrium, change in entropy (dS) is zero. This implies $\frac{\partial S_A}{\partial E_A} = \frac{\partial S_B}{\partial E_B}$ at equilibrium. This $\frac{\partial S_A}{\partial E_A}$ is defined as the inverse of the temperature of system A and is denoted as $\beta_A (= 1/T_A)$. This is very well known fact and this definition rightly denotes the fact that heat flows from hotter body to colder body.

Now we allow these two systems to share their electrical charge as well. Thus total entropy (S) is given as, $S = S_A(E_A, Q_A) + S_B(E_B, Q_B)$. Suppose thermal equilibrium is reached between these two systems and hence change in entropy $(dS) = \left(\frac{\partial S_A}{\partial Q_A} - \frac{\partial S_B}{\partial Q_B}\right) dQ_A$. We know as long as thermodynamic equilibrium does not reach, dS is positive. We also know that charge flows from higher potential to lower potential. Let A be the system with higher potential (Φ_A) and hence dQ_A is positive. Thus we can define, $\frac{\partial S_A}{\partial Q_A} \equiv \frac{\Phi_A}{T_A}$. It is easy to see that this definition matches dimensionally as well. Thus equality of $\frac{\Phi}{T}$, equivalently $\beta\Phi$, between two systems indicates their electrical equilibrium.

Now if we allow these two systems to share their angular momentum, we can similarly show that rotational equilibrium is achieved when $\frac{\partial S_A}{\partial J_A}$ equals to $\frac{\partial S_B}{\partial J_B}$. In fact we can define $\frac{\partial S_A}{\partial J_A} \equiv \frac{\Omega_A}{T_A}$. Here Ω_A denotes the angular speed for system A. This definition is not only correct dimensionally, but it also obeys our known observation that angular momentum shifts from rotating object with higher angular speed to rotating object of lower angular speed.

In this discussion, E_A, S_A, T_A, Q_A, Φ_A, J_A and Ω_A, respectively, denotes the energy, entropy, temperature, electric charge, electric potential, angular momentum and rotational speed of system A. Similarly these quantities with subscript B denotes respectively the same for system B.

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